Structure theorems for game trees

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Kohlberg and Mertens [Kohlberg, E. & Mertens, J. (1986) *Econometrica* 54, 1003–1039] proved that the graph of the Nash equilibrium correspondence is homeomorphic to its domain when the domain is the space of payoffs in normal-form games. A counterexample disproves the analog for the equilibrium outcome correspondence over the space of payoffs in extensive-form games, but we prove an analog when the space of behavior strategies is perturbed so that every path in the game tree has nonzero probability. Without such perturbations, the graph is the closure of the union of a finite collection of its subsets, each diffeomorphic to a corresponding path-connected open subset of the space of payoffs. As an application, we construct an algorithm for computing equilibria of an extensive-form game with a perturbed strategy space, and thus approximate equilibria of the unperturbed game.

game theory | extensive form | computing equilibria

1. Introduction

The most useful general result in game theory is the structure theorem of Kohlberg and Mertens (1). It characterizes the topological structure of the graph of the Nash equilibrium correspondence over the space of payoffs from finite normal-form games. Its proof is very simple, yet its corollaries include the existence of equilibria and strategically stable sets of equilibria, and the resulting index theory shows that a component with nonzero index contains a strategically stable set and that an equilibrium with negative index is typically dynamically unstable. Its practical applications include efficient algorithms for computing equilibria of games in normal form. Here we develop analogs for the graph of equilibrium outcomes over a domain that is the space of payoffs at final nodes of a game tree with perfect recall. As for normal-form games, these results yield an algorithm for computing equilibria of games in extensive form.

Kohlberg and Mertens (1) prove that the equilibrium graph is homeomorphic to the space of players' payoffs from pure-strategy profiles of the normal form. For a fixed game tree, however, the equilibrium graph cannot be homeomorphic to the space of payoffs at final nodes, because such games typically have continua of equilibria. Nevertheless, Kreps and Wilson (2) prove that a generic extensive-form game has finitely many equilibrium components, and all equilibria in each component have the same outcome, i.e., the same probability distribution on final nodes. Conceivably the analog of Kohlberg–Merten's characterization might be true for the graph of equilibrium outcomes, but we show in Section 3 that this analog is false. We establish two weaker characterizations. In Section 4, for a partition of the graph corresponding to the subtrees of equilibrium paths, we construct a homeomorphism from each part into a subset of the space of games; the graph is then the closure of the union of these parts. In Section 5 we prove an exact analog when the tree has no zero-probability events. This result applies to any construction based on perturbing players' simplices of mixed strategies, e.g., the graphs of ε-perfect and ε-proper equilibria. Section 6 develops an efficient algorithm for computing equilibria of extensive-form games.

2. Formulation

We consider the space $\mathcal{G}$ of extensive-form games obtained by assigning payoffs to the final nodes of a finite game tree $\Gamma \equiv (T, \prec, U, N, P_*)$ with perfect recall. T is the set of nodes and $\prec$ is the irreflexive binary relation of precedence in the tree $(T, \prec)$; that is, $\prec$ is acyclic and totally orders the predecessors $\{t' \prec t\}$ of t. The subset of final nodes (those with no successors) is $Z \subset T$, U is the partition of $T\backslash Z$ into information sets of the players and nature, N is the set of players, and $P_*(z) > 0$ is the probability that nature's actions do not exclude the final node z. $U_n \subset U$ is the collection of information sets for player $n \in N$ and $A_n(u)$ is n's set of actions (branches of the tree) available at his information set $u \in U_n$. Let $A_n \equiv \cup_u A_n(u)$ be the entire set of n's actions, each labeled differently. Write $u \prec z$, or equivalently $z \succ u$, if $t \prec z \in Z$ for some node $t \in u$, and write $(u, i) \prec z$ if $t \prec t' \preceq z$ for some node t' that follows node $t \in u$ and action $i \in A_n(u)$. Similarly $i \prec i'$ if i, i' are actions at u, u' with $(u, i) \prec u'$. Perfect recall implies that each $(U_n, \prec)$ is a tree. Player n's set of pure strategies is $S_n \equiv \{s : U_n \to A_n | s(u) \in A_n(u)\}$, and his simplex of mixed strategies is $\Sigma_n \equiv \Delta(S_n)$. Kuhn (3) shows that in a game tree with perfect recall each player n can implement a mixture of pure strategies by a payoff-equivalent behavior strategy $b_n = (b_n(u))_{u \in U_n}$ in which each $b_n(u) \in \Delta(A_n(u))$ is a mixture of actions in $A_n(u)$; i.e., $b_n(i|u)$ is the conditional probability at u that n chooses i.

For the fixed tree Γ, the space of games is $\mathcal{G} = \Re^{N \times Z}$, where a game $G \in \mathcal{G}$ assigns payoff $G_n(z)$ to player n at final node z. The space of outcomes is $\Omega = \Delta(Z)$, where an outcome $P \in \Omega$ assigns probability $P(z)$ to z. Let $\mathcal{E} \subset \mathcal{G} \times \Omega$ be the graph of pairs (G, P) for which P is the outcome induced by P_* and an equilibrium of the game G.

3. A Counterexample

A simple example shows that the analog of Kohlberg-Mertens' structure theorem cannot be true for every game tree. Consider the tree in which player 1 chooses an action in $A_1 = \{T, B\}$ and the game ends if 1 chooses T; otherwise, knowing that 1 chose B, 2 chooses an action in $A_2 = \{L, R\}$. For this tree, the projection $p : \mathcal{E} \to \mathcal{G}$, $p(G, P) = G$, is a proper map; i.e., the inverse image of a compact set is compact. Therefore, if there exists a homeomorphism $\mathcal{H} : \mathcal{E} \to \mathcal{G}$, then the composite map $p \circ \mathcal{H}^{-1} : \mathcal{G} \to \mathcal{G}$ is also proper, so its local degree is the same at every game in $\mathcal{G}$; see Dold (4) Section VIII.4.4-5. To establish a contradiction, it suffices to present examples at which $p \circ \mathcal{H}^{-1}$ has different local degrees. Consider the games (in normal form)

$$G_{xy} = \begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 2, 2 & 2, 2 \\ x, 3 & y, 1 \end{pmatrix} \end{array}.$$

The game G_{33} has a unique equilibrium path BL that persists in a neighborhood of G_{33}, because B remains a strictly dominant strategy for player 1 and L remains the unique best reply for 2. Therefore, the local degree of $p \circ \mathcal{H}^{-1}$ at G_{33} must be $+1$ or -1. The game G_{31} has the two equilibrium paths BL and T, and again all games in a neighborhood of G_{31} have these same two outcomes. Therefore the local degree of $p \circ \mathcal{H}^{-1}$ at G_{31} must be -2, 0, or $+2$. This contradiction does not occur over the space of normal-form games. For instance, in that larger space, the local degree of $p \circ \mathcal{H}^{-1}$ at G_{31} is $+1$: the index of the outcome

Abbreviation: GNM, Global Newton Method.

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T is 0 and the index of BL is $+1$, and the local degree at G_{31} is the sum of these indices. Similar counterexamples can be constructed for the graphs of subgame-perfect and sequential equilibrium outcomes. Further analysis reveals two ways that the manifold property of the space of games on the tree is not inherited by the graph of equilibrium outcomes. First, (G_{32}, T) has a neighborhood in $\mathcal{E}$ that is a manifold *with boundary.* Second, (G_{21}, T) is a point where the graph bifurcates over the segment $2 - \varepsilon < x < 2 + \varepsilon$: if $x < 2$, then T is the unique equilibrium outcome, at $x = 2$ there is a continuum of equilibrium outcomes, and if $x > 2$, then both T and BL are equilibrium outcomes.

4. A Partial Structure Theorem for Games in Extensive Form

In this section, we prove a structure theorem for graphs of equilibrium outcomes on subtrees. For this section only, we assume that the space of games in $\mathcal{G}_+ \equiv \Re_{++}^{N\times Z}$. This assumption is without loss of generality, because there exists a homeomorphism between $\mathcal{G}$ and $\mathcal{G}_+$ that preserves the equilibrium correspondence. Indeed, there exists a monotonic function $f : \Re \to \Re_{++}$ such that $f(x)$ goes to zero as x goes to $-\infty$, and $f(x)$ goes to $+\infty$ as x goes to $+\infty$. Then the map that sends $G \in \mathcal{G}$ to $G' \in \mathcal{G}_+$ given by $G'_n(z) = G_n(z) + f(\lambda_n) - \lambda_n$, where $\lambda_n = \min_z G_n(z)$, is a homeomorphism that preserves the equilibrium correspondence. It suffices therefore to use the graph $\bar{\mathcal{E}}$ of equilibrium outcomes over $\mathcal{G}_+$. Consider first the subset $\mathcal{E}_+ \equiv \{(G, P) \in \bar{\mathcal{E}} | P(z) > 0\ \forall z\}$ of the graph over $\mathcal{G}_+$, where all nodes have positive probability. Define the map $\Phi : \mathcal{E}_+ \to \mathcal{G}_+$, $\Phi(G, P) = H$, by $H_n(z) = G_n(z)P(z)$.

Lemma 4.1. *$\Phi : \mathcal{E}_+ \to \mathcal{G}_+$ is a homeomorphism.*

Proof: We construct a continuous map $\Psi : \mathcal{G}_+ \to \mathcal{E}_+$ and show that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are identities on $\mathcal{E}_+$ and $\mathcal{G}_+$, respectively, which immediately implies the result.

Define $\Psi : \mathcal{G}_+ \to \mathcal{E}_+$ as follows. Given any $H \in \mathcal{G}_+$, first define $H_n(u, i) \equiv \Sigma_{z\succ(u,i)} H_n(z)$ and $H_n(u) \equiv \Sigma_{i\in A_n(u)} H_n(u, i)$; next, define $b_n(i|u) \equiv H_n(u, i)/H_n(u)$; then the outcome P is obtained from P_* and the behavior profile b; and finally, define $G_n(z) \equiv H_n(z)/P(z)$. Each of these is positive and $G \in \mathcal{G}_+$ as required. The outcome P induces probabilities $P(u)$ and $P(u, i)$ of the events that the information set u is reached and that action i is taken there; and because $P(u, i) > 0$, also the conditional probability $Q(z|u, i) = P(z)/P(u, i)$ if $z \succ (u, i)$. Thus

$$H_n(u, i) = \sum_{z\succ(u,i)} H_n(z) = \sum_{z\succ(u,i)} G_n(z)P(z)$$

$$= G_n(u, i)P(u, i) = G_n(u, i)P(u)b_n(i|u)$$

and $$H_n(u) = \sum_{i\in A_n(u)} H_n(u, i) = G_n(u)P(u)$$

where $$G_n(u, i) \equiv \sum_{z\succ(u,i)} G_n(z)Q(z|u, i)$$

and $$G_n(u) \equiv \sum_{i\in A_n(u)} G_n(u, i)b_n(i|u).$$

Therefore

$$b_n(i|u) \equiv H_n(u, i)/H_n(u) = [G_n(u, i)/G_n(u)]b_n(i|u),$$

which implies that $G_n(u, i) = G_n(u)$ for each $i \in A_n(u)$. From its definition above, $G_n(u, i)$ is n's continuation value from action i at u in the game G with behavior profile b, so the fact that it is the same for each action $i \in A_n(u)$ verifies that b is an equilibrium. Therefore, $(G, P) \in \mathcal{E}_+$ as required for Ψ to be well defined. Obviously, Ψ is a continuous map. Also, by construction, $\Phi \circ \Psi$ is the identity on G_+. To complete the proof, it remains to be shown that $\Psi \circ \Phi$ is the identity on $\mathcal{E}_+$. To prove this, suppose $(G', P') \in \mathcal{E}_+$ and let $v'_n(i|u)$ be n's equilibrium continuation value from i at u. Because $P' \gg 0$, all actions at u are optimal, so $v'_n(i|u) = v'_n(u)$ for all $i \in A_n(u)$, and $\Sigma_{z\succ(u,i)} G'_n(z)P'(z) = v'_n(u)P'(u)b'_n(i|u)$, where b' is the behavior profile for P'. Therefore, if $\Phi(G', P') = H$ and $\Psi(H) = (G, P)$ as above, then $b'_n(i|u) = H_n(u, i)/H_n(u) \equiv b_n(i|u)$. Because P' is uniquely determined by b', $P' = P$; and because $G'_n(z)P'(z) = H_n(z) = G_n(z)P(z)$, also $G' = G$. Thus $\Psi \circ \Phi$ is the identity, as asserted. □

Each equilibrium induces an equilibrium with full support on the pruned tree obtained by eliminating all nodes following a branch with zero probability. Therefore:

Theorem 4.2. *There exist finitely many subsets $\mathcal{E}_k \subset \bar{\mathcal{E}}$, such that (a) each $\mathcal{E}_k$ is homeomorphic to an open path-connected subset of $\mathcal{G}_+$ and (b) $\bar{\mathcal{E}}$ is the closure of $\cup_k \mathcal{E}_k$.*

Proof: Each $\mathcal{E}_k$ is the subset of the graph consisting of all pairs $(G, P) \in \bar{\mathcal{E}}$ in which the support of the outcome P is $Z_k \subset Z$ and the outcome is induced by an equilibrium in which any pure strategy is strictly inferior if it does not exclude some information set on the equilibrium path and uses there an action that has zero probability in the equilibrium. It is then clear that these $\mathcal{E}_k$s satisfy part b, so it remains to prove a. Let $\mathcal{G}_k$ be the projection of $\mathcal{G}_+$ onto the coordinates corresponding to Z_k. $\mathcal{G}_k$ is the space of games with the tree Γ_k obtained from Γ by retaining only branches leading to nodes in Z_k. Let $\hat{\mathcal{E}}_k$ be the graph of the (completely mixed) equilibrium outcome correspondence over $\mathcal{G}_k$. By *Lemma 4.1*, $\hat{\mathcal{E}}_k$ is homeomorphic to $\mathcal{G}_k$. Express $\mathcal{G}_+$ as the product of $\mathcal{G}_k$ with $\mathcal{F}_k = \Re_{++}^{N\times Z\setminus Z_k}$. Then $\hat{\mathcal{E}}_k \times \mathcal{F}_k$ is homeomorphic to $\mathcal{G}_k \times \mathcal{F}_k$, using the identity map on the second factor $\mathcal{F}_k$. $\mathcal{E}_k$ is an open subset of $\hat{\mathcal{E}}_k \times \mathcal{F}_k$ and is therefore mapped onto an open subset of $\mathcal{G}_+$ by the homeomorphism. It remains to show that $\mathcal{E}_k$ is path-connected. Let $((G_k, F_k), P)$ and $((G'_k, F'_k), P')$ be two points in $\mathcal{E}_k$. Because $\hat{\mathcal{E}}_k$ is homeomorphic to $\mathcal{G}_k$, connect (G_k, P_k) and (G'_k, P'_k) by a path in $\hat{\mathcal{E}}_k$. Let $F^*_k \in \mathcal{F}_k$ be a vector whose coordinates are all strictly less than the payoff received by any player at any equilibrium along this path, and less in each coordinate than F_k and F'_k. Then the linear paths connecting $((G_k, F_k), P)$ to $((G_k, F^*_k), P)$ and $((G'_k, F'_k), P')$ to $((G'_k, F^*_k), P')$ are in $\mathcal{E}_k$, because decreasing payoffs to players at nodes in $Z\setminus Z_k$ has no effect on equilibria. Now the points $((G_k, F^*_k), P)$ and $((G'_k, F^*_k), P')$ can be path-connected using the path in $\hat{\mathcal{E}}_k$. The choice of F^*_k ensures that the path belongs to $\mathcal{E}_k$. □

The counterexample in Section 3 shows that, for some trees, a violation of the manifold property occurs at boundaries among the open subsets $\mathcal{E}_k$. These boundaries lie within the lower-dimensional set of nongeneric games excluded by theorems showing that a generic extensive-form game has a finite number of equilibrium outcomes; see Kreps and Wilson (2) or Govindan and Wilson (5).

5. A Structure Theorem for Games with Perturbed Strategies

In this section, we assume that each player n's simplex Σ_n of mixed strategies is perturbed to the compact convex subset $\Sigma_n^\varepsilon \subset \Sigma_n$ disjoint from the boundary. Because $P_* \gg 0$ by assumption, these perturbations assure that every mixed-strategy profile in $\Sigma^\varepsilon \equiv \Pi_n \Sigma_n^\varepsilon$ yields a positive probability for every final node.

From a mixed-strategy profile σ, one can derive the corresponding "nonexclusion" or *enabling* profile $p \in \Pi_n[0, 1]^{L_n}$ as follows. L_n is the set of player n's last actions; that is, $i \in L_n \subset A_n$ iff there exists $z \in Z$ such that i is the $\prec$-maximal element in $A_n(z) \equiv \{i' \in A_n | i' \prec z\}$; that is, $i = \ell_n(z) \equiv \text{Arg max } A_n(z)$. If $L_n = \emptyset$, then n is a dummy player so p_n can be omitted from the profile. For each $i \in L_n$, $p_n(i)$ is the probability under σ_n that n's selected pure strategy does not exclude i or any of n's actions preceding i. One computes $p_n(i)$ as follows. The subset of n's pure strategies that do not exclude z is $S_n(z) \equiv \{s \in S_n | (u, i) \prec$

$z \Rightarrow s(u) = i\}$. If n uses the mixed strategy σ_n, then the probability that n does not exclude z is $P_n(z) = \Sigma_{s \in S_n(z)} \sigma_n(s)$, or $P_n(z) = 1$ if $A_n(z) = \emptyset$. Let $Z_n(i) \equiv \ell_n^{-1}(i) = \{z | \ell_n(z) = i\}$ and let $s_n(i) \equiv S_n(z)$ for each $z \in Z_n(i)$ be the event that n's pure strategy enables $i \in L_n$. Then $p_n(i) \equiv \text{Prob}_\sigma(s_n(i)) = P_n(z)$ for each $z \in Z_n(i)$.

The feasible set of enabling profiles is $\mathcal{C} \equiv \Pi_n C_n$, where for each nondummy player n, his feasible set of enabling strategies is

$$C_n = \{p_n \in [0,1]^{L_n} | (\exists\, \sigma_n \in \Sigma_n^\varepsilon)(\forall\, i \in L_n) p_n(i) = \sum_{s \in s_n(i)} \sigma_n(s)\}.$$

Observe that C_n is compact and convex. Because Σ_n^ε is disjoint from the boundary of Σ_n, $p_n \in C_n$ only if each $p_n(i) > 0$.

For a mixed strategy $\sigma_n \in \Sigma_n^\varepsilon$, the induced enabling strategy p_n is equivalent to a behavior strategy b_n obtained as follows. Each $b_n(i|u)$ is proportional to $\beta_n(u, i) \equiv \Sigma \sigma_n(s)$, where the sum is over those pure strategies $s \in S_n$ such that if $(u', i') \preceq (u, i)$, then $s(u') = i'$. Each pure strategy in this sum selects an action at each of n's next information sets after (u, i), if any. Therefore, if (u', i') is the immediate predecessor of u among n's information sets, then $\beta_n(u', i') = \Sigma_{i \in A_n(u)} \beta_n(u, i)$. This recursion enables calculation of β_n by working backward from n's final information sets where $\beta_n(u, i) = p_n(i)$. Conversely, from a behavior strategy one can derive the enabling strategy via $p_n(i) = \Pi_{(u', i') \preceq (u,i)} b_n(i'|u')$ for each $i \in L_n$, because the normalizing factors cancel along a path. Similarly, an enabling profile p yields an outcome P via $P(z) = P_*(z) \times \Pi_n P_n(z)$, and an outcome implies the behavior strategy via $b_n(i|u) \propto \Sigma_{z \succ (u,i)} P(z)$.

An enabling strategy is the minimal representation of a behavior strategy that preserves the linearity and convexity of the space of mixed strategies. This representation avoids complications from auxiliary constraints imposed by non-minimal representations. S. Elmes (Columbia D.P. 490, 1990, personal communication) notes these complications in her repair of defects in appendix C of ref. 1. Enabling strategies are closely related and essentially equivalent to strategies in sequence form [von Stengel (6)].

Given an outcome P, the expected payoff of player n can be written

$$\sum_{z \in Z} G_n(z)P(z) = \sum_{i \in L_n} p_n(i) \sum_{z \in Z_n(i)} G_n(z)P^n(z) + \sum_{z | A_n(z) = \emptyset} G_n(z)P^n(z)$$
$$= \sum_{i \in L_n} p_n(i) v_n(i) + v_n(\emptyset),$$

where $P^n(z) \equiv P_*(z) \times \Pi_{n' \neq n} P_{n'}(z)$, $v_n(\emptyset) \equiv \Sigma_{z|A_n(z)=\emptyset} G_n(z)P^n(z)$, and $v_n(i) \equiv \Sigma_{z \in Z_n(i)} G_n(z)P^n(z)$. As in Gül, Pearce, and Stacchetti (7), let $r_n : \Re^{L_n} \to C_n$ be the retraction that maps x to the point in C_n closest to x in Euclidean distance; namely, $p_n = r_n(x)$ iff $\Sigma_{i \in L_n}[p_n'(i) - p_n(i)][x(i) - p_n(i)] \le 0$ for all $p_n' \in C_n$.

Lemma 5.1. *An enabling strategy $p_n \in C_n$ for a nondummy player n is an optimal reply to $\sigma \in \Sigma^\varepsilon$ (or an equivalent profile of behavior or enabling strategies) iff $p_n = r_n(p_n + v_n)$.*

Proof: A mixed strategy $\sigma_n \in \Sigma_n^\varepsilon$ is optimal for player n iff for all $\sigma_n' \in \Sigma_n^\varepsilon$

$$0 \le \sum_{i \in L_n} \sum_{s \in s_n(i)} [\sigma_n(s) - \sigma_n'(s)] v_n(i) \equiv \sum_{i \in L_n} [p_n(i) - p_n'(i)] v_n(i),$$

where $p_n(i) \equiv \Sigma_{s \in s_n(i)} \sigma_n(s)$ and $p_n'(i) \equiv \Sigma_{s \in s_n(i)} \sigma_n'(s)$ are the corresponding enabling strategies in C_n. Because the possible values of p_n' include all of C_n, this is precisely the variational inequality that characterizes the equality $p_n = r_n(p_n + v_n)$. □

Thus in terms of enabling strategies, an equilibrium is a fixed point $p = r(p + v)$ in the sense that $p_n = r_n(p_n + v_n)$ for each player n, where v is derived from p as above. Hereafter, we consider only equilibria in enabling strategies. As above the set of enabling profiles is $\mathcal{C}$. Represent a game as a point $G \in \mathcal{G} \equiv \Re^{N \times Z}$, the space of players' payoffs at final nodes. Let $\mathcal{E} \subset \mathcal{G} \times \mathcal{C}$ be the graph of the equilibrium correspondence over the space of games for the game tree Γ. Let $E[\cdot|\cdot]$ be the conditional expectation operator for a fixed strictly positive probability distribution in $\Delta(Z)$. Define the map $\mathcal{H} : \mathcal{E} \to \mathcal{G}$, $\mathcal{H}(G, p) = H$, by

$$H_n(z) = G_n(z) - g_n(\ell_n(z)) + p_n(\ell_n(z)) + v_n(\ell_n(z)),$$

or $H_n(z) = G_n(z)$ if $A_n(z) = \emptyset$, for each player n and final node z, where $g_n(i) = E[G_n(z)|Z_n(i)]$ for each $i \in L_n$.

Theorem 5.2. *$\mathcal{H}$ is a homeomorphism.*

Proof: Define $\mathcal{K} : \mathcal{G} \to \mathcal{E}$ as follows. Given $H \in \mathcal{G}$, first let $h_n(i) = E[H_n(z)|Z_n(i)]$ for each n and $i \in L_n$. Also, let $p_n = r_n(h_n)$ and $v_n = h_n - p_n$ for each nondummy player n. Next, let $g_n(i) = h_n(i) + [v_n(i) - \Sigma_{z \in Z_n(i)} H_n(z)P^n(z)]/P^n(Z_n(i))$; and finally, let $G_n(z) = H_n(z) + g_n(\ell_n(z)) - h_n(\ell_n(z))$, or $G_n(z) = H_n(z)$ if $A_n(z) = \emptyset$. The G thus constructed satisfies $\Sigma_{z \in Z_n(i)} G_n(z)P^n(z) = v_n(i)$ for $i \in L_n$. Therefore, $v_n(i)$ is indeed n's marginal expected payoff from increasing $p_n(i)$, which by *Lemma 5.1* is sufficient for p_n to be an optimal reply by a nondummy player n in the game G. Thus, $\mathcal{K}$ is a well defined continuous map. It is immediate from our construction that $\mathcal{H} \circ \mathcal{K}$ is the identity on $\mathcal{G}$. We will now show that $\mathcal{K} \circ \mathcal{H}$ is the identity on $\mathcal{E}$, which then implies that $\mathcal{K} = \mathcal{H}^{-1}$, i.e., $\mathcal{H}$ is a homeomorphism. Suppose $H = \mathcal{H}(G', p')$ and $(G, p) = \mathcal{K}(H)$. For each n and $i \in L_n$, $h_n(i) \equiv E[H_n(z)|Z_n(i)] = p_n'(i) + v_n'(i)$. Because $p_n \equiv r_n(h_n)$, we therefore have that $p_n(i) = p_n'(i)$ and also that $v_n(i) = v_n'(i)$ for all n and $i \in L_n$. By the definition of v_n and $\mathcal{H}$,

$$v_n(i) - g_n'(i)P^n(Z_n(i)) = \sum_{z \in Z_n(i)} [G_n'(z) - g_n'(i)]P^n(z)$$
$$= \sum_{z \in Z_n(i)} [H_n(z) - p_n(i) - v_n(i)]P^n(z)$$
$$= \sum_{z \in Z_n(i)} [H_n(z) - h_n(i)]P^n(z).$$

Hence, $g_n'(i) = h_n(i) + [v_n(i) - \Sigma_{z \in Z_n(i)} H_n(z)P^n(z)]/P^n(Z_n(i))$, which by the definition of $\mathcal{K}$ is $g_n(i)$. Consequently, $G = G'$, and $\mathcal{K} \circ \mathcal{H}$ is the identity on $\mathcal{E}$. □

A repetition of the proof in ref. 1 shows that H extends to a homeomorphism between the one-point compactifications of $\mathcal{E}$ and $\mathcal{G}$ and that $\text{proj}_\mathcal{G} \circ \mathcal{H}^{-1}$ is linearly homotopic to the identity map on the one-point compactification of $\mathcal{G}$; thus, $\text{proj}_\mathcal{G}$ is a map of degree one. An obvious corollary is that their theorem applies to Nash equilibria of normal-form games with perturbed sets of mixed strategies. *Theorem 5.2* applies to stronger definitions of equilibrium in extensive-form games based on perturbed strategy sets. For example, if $\Sigma_n^\varepsilon = \{\varepsilon\bar{\sigma} + [1 - \varepsilon]\sigma | \sigma \in \Sigma_n)\}$, where $\varepsilon > 0$ and $\bar{\sigma}$ is the barycenter of Σ_n, then $\mathcal{E}$ is the graph of ε-perfect equilibria over the space of extensive-form games on the tree Γ. Similarly, if Σ_n^ε is the convex hull of the points generated by all permutations of the coordinates of the vector $(1, \varepsilon, \varepsilon^2, \ldots)$, rescaled to lie in Σ_n, then $\mathcal{E}$ is the graph of ε-proper equilibria; see ref. 1, proposition 5. In other applications where each r_n is smooth because Σ_n^ε has a smooth boundary, $\mathcal{H}$ is a diffeomorphism.

6. An Algorithm for Computing Equilibria of Perturbed Extensive-Form Games

In this section, we apply *Theorem 5.2* to construct an algorithm for computing equilibria of an extensive-form game defined on a tree Γ with perturbed sets of mixed strategies. This algorithm

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is a variant of the algorithms in Govindan and Wilson (8, 9) for normal-form games; see those articles for technical background, detailed specifications, computer programs, and numerical results for N-player games. Proofs in those articles apply here almost verbatim: the only difference is that the retraction r_n to player n's simplex Σ_n of mixed strategies is replaced here by the retraction to the polytope C_n of n's enabling strategies.

First, we describe a general parametric method that exploits the key simplifying feature that the retraction r is independent of the payoffs. Represent the game G as the pair $(\tilde{G}, g)$ where $\tilde{G}_n(z) = G_n(z) - g_n(\ell_n(z))$ using $g_n \in \Re^{L_n}$ as defined in Section 5. Using the homeomorphism $\mathcal{H}$ of *Theorem 5.2*, the algorithm finds an equilibrium of G by tracing the solutions of the equation $\mathcal{H}(\tilde{G}, g + \lambda\gamma; p(h)) = H \equiv (\tilde{G}, h)$, where $\lambda\gamma$ parameterizes a ray whose origin represents the game $(\tilde{G}, g)$ whose equilibria are to be computed. As in *Theorem 5.2*, at each solution h, the equilibrium $p(h)$ of the game $(\tilde{G}, g + \lambda\gamma)$ is the enabling profile $p(h) = r(h)$ obtained by retracting h to C. The algorithm starts by choosing a ray γ and an initial scalar parameter $\lambda°$ sufficiently large that the game $(\tilde{G}, g + \lambda°\gamma)$ has a unique equilibrium $p°$. One then follows the trajectory in the graph above the line segment through the two points $(\tilde{G}, g + \lambda°\gamma)$ and $(\tilde{G}, g)$ as the parameter λ decreases to zero. The implementation must contend with the usual two complications of homotopy methods: (*i*) the ray γ must be generic to ensure that the equilibrium outcome $p°$ is unique when $\lambda°$ is sufficiently large and to exclude bifurcations along the trajectory; and (*ii*) the trajectory in the graph includes reversals of orientation, so the parameter λ cannot decrease monotonically. Each time the trajectory crosses $\lambda = 0$ yields an additional equilibrium of the game $(\tilde{G}, g)$, and for generic games these equilibria have alternating indices $+1$ and -1. Different choices of the ray γ can yield different equilibria.

The parametric method can be implemented via the Global Newton Method (GNM) of Smale (10). Using a monotonic time parameter t, the trajectory is $(h(t), \lambda(t))_{t \geq t°}$. The algorithm starts at $\lambda(t°) = \lambda°$ and finds the kth equilibrium on the trajectory at a time $t^k > t^{k-1}$ for which $\lambda(t^k) = 0$. Actual computations use discrete steps, but here time is assumed to be continuous. In its simplest form, GNM finds a root of a differentiable function F by tracing the trajectory of the differential equation $\dot{h} = -\theta(h)[DF(h)]^{-1} \cdot F(h)$ starting from an initial point $h°$. In this form, $\theta(h)$ is a continuous scalar velocity, and $DF(h)$ is the Jacobian matrix of F at h. However, better numerical properties are obtained by starting from a unique solution $h°$ to $F(h°) = \lambda°\gamma$ and using the equations $\dot{h} = -(\mathrm{Adj}[DF(h)]) \cdot \gamma$ and $\dot{\lambda} = -\mathrm{Det}[DF(h)]$ corresponding to the velocity $\theta(h) = -\dot{\lambda}/\lambda$. In particular, replacing the inverse by the adjoint matrix of the Jacobian enables the trajectory to pass through singularities of codimension 1. Note that the trajectory reverses orientation each time the determinant changes sign.

GNM is invoked by translating the equations in the proof of *Theorem 5.2*. Assuming there are no dummy players, let $F : \Pi_n \Re^{L_n} \to \Pi_n \Re^{L_n}$ be the displacement map of $\mathcal{H}$ with $\tilde{G}$ fixed; that is,

$$F_n(i)(h) = h_n(i) - g_n(i) - p_n(i) - \sum_{z \in Z_n(i)} \tilde{G}_n(z) P^n(z)$$

for each $i \in L_n$, where from $p = r(h)$ one constructs the outcome P. Then an equilibrium of the game $(\tilde{G}, g + \lambda\gamma)$ is obtained from a solution of the equation $F(h) = \lambda\gamma$. In vector form, let $F(h) = h - g - p - \tilde{G} \cdot Q(p)$, where in the matrix $Q(p)$ a nonzero element $Q_{zi} = P^n(z)$ if $z \in Z_n(i)$ and $i \in L_n$. The Jacobian DF at h is $DF(h) = I - [I - \tilde{G} \cdot DQ(p)] \cdot Dr(h)$, where I is the identity matrix, DQ is the Jacobian of Q at $p = r(h)$, and Dr is the Jacobian of r at h. The resulting trajectory of GNM is the path of the differential equation

$$(\dot{h}, \dot{\lambda}) = -([\mathrm{Adj}[DF(h)]] \cdot \gamma, \mathrm{Det}[DF(h)]).$$

Everywhere on this trajectory, $\mathcal{H}(\tilde{G}, g + \lambda\gamma; p(h)) = (\tilde{G}, h)$ if one starts with $F(h°) = \lambda°\gamma$, where for a generic ray γ, $\lambda(t°) = \lambda°$ is sufficiently large that $p° = r(h°)$ is the unique equilibrium of the game $(\tilde{G}, g + \lambda°\gamma)$. The Jacobian DF is continuous except that, when C is polyhedral (as in important applications), Dr changes discontinuously where some p_n moves from one face to another of C_n. We show in ref. 9 that the trajectory is continuous across such a boundary even though its direction changes discontinuously. For standard applications such as ε-perfect equilibria (used to approximate sequential equilibria), the block-diagonal matrix Dr is constant on each face of the polyhedron C.

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1. Kohlberg, E. & Mertens, J. (1986) *Econometrica* **54,** 1003–1039.
2. Kreps, D. & Wilson, R. (1982) *Econometrica* **50,** 863–894.
3. Kuhn, H. (1953) in *Contributions to the Theory of Games* **II,** eds. Kuhn, H. & Tucker, A. (Princeton Univ. Press, Princeton, NJ), pp. 193–216.
4. Dold, A. (1972) *Lectures on Algebraic Topology* (Springer, New York).
5. Govindan, S. & Wilson, R. (2001) *Econometrica* **69,** 765–769.
6. von Stengel, B. (1996) *Games Econ. Behav.* **14,** 220–246.
7. Gül, F., Pearce, D. & Stacchetti, E. (1993) *Math. Oper. Res.* **18,** 548–552.
8. Govindan, S. & Wilson, R. (2002) *J. Econ. Dynam. Control* **26,** in press.
9. Govindan, S. & Wilson, R. (2002) *J. Econ. Theory* **104,** in press.
10. Smale, S. (1976) *J. Math. Econ.* **3,** 107–120.