JOURNAL OF ECONOMIC THEORY **68**, 258–265 (1996)
ARTICLE NO. 0014

Fictitious Play Property for Games with Identical Interests*

DOV MONDERER†

Faculty of Industrial Engineering and Management, The Technion, Haifa 32000, Israel

AND

LLOYD S. SHAPLEY

Department of Economics and Department of Mathematics, University of California, Los Angeles, California 90024

Received October 26, 1993; revised October 24, 1994

An n-person game has identical interests if it is best response equivalent in mixed strategies to a game with identical payoff functions. It is proved that every such game has the fictitious play property. *Journal of Economic Literature* Classification Numbers: C72, C73. © 1996 Academic Press, Inc.

INTRODUCTION

Consider n players, engaged in a repeated play of a finite game in strategic (normal) form. Every player assumes that each of the other players is using a stationary (i.e., time independent) mixed strategy. The players observe the actions taken in previous stages, update their beliefs about their opponents' strategies, and choose myopic pure best responses against these beliefs. In a "Fictitious Play," proposed by Brown [1], every

* We thank Vijay Krishna, Aner Sela, and the anonymous referees for very helpful remarks. Part of this research was done when the first author was visiting the Department of Economics, Queen's University, Kingston, Canada. This work was supported by the Fund for the Promotion of Research in the Technion. Some of the results in this paper were previously contained in the manuscript "Potential Games."

† E-mail: dov@techunix.technion.ac.il.

0022-0531/96 $18.00

player (except for Player i) takes the empirical distribution of Player i's actions to be his belief about Player i's mixed strategy.[1]

The definition of the fictitious play process may depend on first move rules, weights assigned to initial beliefs, and tie breaking rules determining the particular best replies chosen at each stage. In this note we stick to the original definition of fictitious play in which the first moves are chosen arbitrarily and no tie-breaking rules are assumed. Our result remains valid under the above-mentioned possible modifications of the definition.

We say that the process converges in beliefs to equilibrium if the sequence of beliefs (regarded as mixed strategies) is as close as we wish to the set of equilibria after a sufficient number of stages.[2] Equivalently, the process converges in beliefs to equilibrium if for every $\varepsilon > 0$, the beliefs are in ε-equilibrium after a sufficient number of stages. We say that a game has the fictitious play property (FPP) if every fictitious play process converges in beliefs to equilibrium. Shapley [11] constructed an example of a 3×3 2-person game without the FPP. It is therefore important to identify classes of games with the FPP.

Robinson [10] proved that every 2-person zero-sum game has the FPP. Miyasawa [6] proved (using a particular tie-breaking rule) that every 2-person 2×2 game has the FPP.[3] Milgrom and Roberts [5] showed that every game which is dominance solvable has the FPP. Krishna [4] proved that if the strategy sets are linearly ordered, then every game with strategic complementarities and diminishing returns has the FPP, if a particular tie-breaking rule is used. Deschamps [3] proved that 2-person linear Cournot games have the FPP. Thorlund–Petersen [12] proved that n-person linear Cournot games have the FPP.[4]

In this note we show that a fictitious play process converges in beliefs to equilibrium if and only if it converges in beliefs to equilibrium in the Cesaro mean. We then show that every game in which all players have the same payoff function has the FPP. Obviously, the FPP is invariant under utility transformations that preserve the mixed best response structure of the game. Consequently, every game which is best response equivalent in mixed strategies to a game with identical payoff functions must have the

[1] Brown's original fictitious play was defined for 2-person games. There are other (and perhaps better) ways to define this process for n-person games; e.g., we can require that every player choose a myopic best response to the empirical joint distribution of the other players' actions.

[2] In this definition, the ith component of the belief sequence is the mixed strategy that is believed to be used by Player i by all other players.

[3] Monderer and Sela [9] show that degenerate 2×2 games in which one and only one of the players has equivalent strategies and the other player does not have weakly dominated strategy do not have the FPP. At the end of this note we show that every nondegenerate 2×2 game has the FPP.

[4] This paper deals with a fictitious play-like process in a larger clas of Cournot games. This process coincides with the standard fictitious play process in the linear model.

FPP. Such games are called *games with identical interests*. We further note in this paper that every nondegenerate 2-person 2×2 game is best response equivalent in mixed strategies either to a game of the form (A, A) or to a zero-sum game (i.e., a game of the form $(A, -A)$). Miyasawa's theorem is thus derived by combining our theorem with Robinson's.[5]

1. Fictitious Play

Let Γ be a finite game in strategic form. The set of players is $N = \{1, 2, ..., n\}$, the set of strategies of Player i is Y^i, and the payoff function of Player i is u^i: $Y \to R$, where $Y = Y^1 \times Y^2 \times ... \times Y^n$ and R denotes the set of real numbers. For $i \in N$ let Δ^i be the set of mixed strategies of Player i. That is,

$$\Delta^i = \left\{ f^i\colon Y^i \to [0, 1]\colon \sum_{y^i \in Y^i} f^i(y^i) = 1 \right\}.$$

We identify the pure strategy $y^i \in Y^i$ with the extreme point of Δ^i which assigns a probability 1 to y^i. Set $\Delta = \mathsf{X}_{i \in N} \Delta^i$. For $i \in N$ let U^i be the payoff function of player i in the mixed extension of Γ. That is,

$$\begin{aligned} U^i(f) &= U^i(f^1, f^2, ..., f^n) \\ &= \sum_{y \in Y} u^i(y^1, y^2, ..., y^n)\, f^1(y^1)\, f^2(y^2) \cdots f^n(y^n) \qquad \text{for all} \quad f \in \Delta. \end{aligned}$$

For $i \in N$ and for $f \in \Delta$ we denote

$$v^i(f) = \max\{ U^i(g^i, f^{-i}) : g^i \in \Delta^i \}.$$

Let $g \in \Delta$, and let $\varepsilon > 0$. g is an *ε-equilibrium* if for each $i \in N$,

$$U^i(g) \geqslant U^i(f^i, g^{-i}) - \varepsilon \qquad \text{for all} \quad f^i \in \Delta^i.$$

Denote by $K = K(\Gamma)$ the equilibrium (in mixed strategies) set of Γ, and denote by $\| \ \|$ any fixed Euclidean norm on Δ. For $\delta > 0$ set

$$B_\delta(K) = \{ g \in \Delta : \min_{f \in K} \|g - f\| < \delta \}.$$

A *path* in Y is a sequence $y = (y(t))_{t=1}^\infty$ of elements of Y. A *belief path* is a sequence $f = (f(t))_{t=1}^\infty$ in Δ. We say that the belief path $(f(t))_{t=1}^\infty$ *converges to equilibrium* if each limit point is an equilibrium point; that is, if

[5] Miyasawa's proof is 35 pages long.

for every $\delta > 0$ there exists an integer T such that $f(t) \in B_\delta(K)$ for all $t \geqslant T$. Obviously, the belief path converges to equilibrium if and only if for every $\varepsilon > 0$ there exists an integer T such that $f(t)$ is an ε-equilibrium for every $t \geqslant T$. We say that the belief path $(f(t))_{t=1}^{\infty}$ *converges to equilibrium in the Cesaro mean* if

$$\lim_{T \to \infty} \frac{\#\{1 \leqslant t \leqslant T : f(t) \notin B_\delta(K)\}}{T} = 0 \qquad \text{for every} \quad \delta > 0.$$

Equivalently, it can be shown that the belief path converges to equilibrium in the Cesaro mean iff for every $\varepsilon > 0$,

$$\lim_{T \to \infty} \frac{\#\{1 \leqslant t \leqslant T : f(t) \notin K_\varepsilon\}}{T} = 0, \tag{1.1}$$

where K_ε denotes the set of all ε-equilibrium points. Obviously, convergence to equilibrium implies convergence to equilibrium in the Cesaro mean.

To each path y we associate a belief path f_y:

$$f_y(t) = \frac{1}{t} \sum_{s=1}^{t} y(t) \qquad \text{for every} \quad t \geqslant 1.$$

Note that

$$f_y(t+1) = f_y(t) + \frac{1}{t+1} (y(t+1) - f_y(t)). \tag{1.2}$$

A path $y = (y(t))_{t=1}^{\infty}$ is a *fictitious play process* if for every $i \in N$

$$v^i(f_y(t)) = U^i(y^i(t+1), f_y^{-i}(t)) \qquad \text{for every} \quad t \geqslant 1. \tag{1.3}$$

Note that (1.3) means that $y^i(t+1)$ is a best response to $f_y^{-i}(t)$. We say that the fictitious play process $(y(t))_{t=1}^{\infty}$ *converges in beliefs to equilibrium* (*in the Cesaro mean*) if the associated belief path converges to equilibrium(in the Cesaro mean).

We say that Γ has the *fictitious play property* (FPP) if every fictitious play process in Γ converges in beliefs to equilibrium.

The next lemma will be used in our main theorem. It may also be useful in identifying other classes of games with the FPP.The proof of this lemma is similar to the proof of the well–known theorem about bounded sequences of real numbers: Convergence in the Cesaro mean of the Cesaro means implies convergence in the Cesaro mean.

LEMMA 1. *For every game in strategic form, a fictitious play process converges in beliefs to equilibrium if and only if it converges in beliefs to equilibrium in the Cesaro mean.*

Proof. Let $y=(y(t))_{t=1}^{\infty}$ be a fictitious play process, and let $f=(f(t))_{t=1}^{\infty}$ be its associated belief process. Obviously we have to prove only the "if" part. Let then $\delta>0$ be given. Denote

$$M=\max_{f,g\in\Delta}\|f-g\|, \tag{1.4}$$

and choose $\eta<\delta/(2\delta+M)$. By (1.1) there exists an integer T_0 such that for every $T\geqslant T_0$,

$$\#\{1\leqslant t\leqslant T: f(t)\notin B_\delta(K)\}<\eta T. \tag{1.5}$$

We claim that for every $T\geqslant T_0$, $f(T)\in B_{2\delta}(K)$.

Suppose $T\geqslant T_0$ and $f(T)\notin B_{2\delta}(K)$. Then it can be easily verified (because the step size at stage t is $1/t$) that $f(t)\notin B_\delta$ for $T\leqslant t\leqslant T+(\delta/(\delta+M))T$. Hence

$$\#\left\{1\leqslant t\leqslant T+\frac{\delta}{\delta+M}T: f(t)\notin B_\delta(K)\right\}\geqslant\frac{\delta}{\delta+M}T>\eta\left(T+\frac{\delta}{\delta+M}T\right),$$

contradicting (1.5). ∎

THEOREM A. *Every game with identical payoff functions has the fictitious play property.*

Proof. Denote the joint payoff function by u. That is, $u^i=u$ for all $i\in N$. Recall that U denotes the multilinear extension of u. Let $(f(t))_{t=0}^{\infty}$ be the belief process associated with a fictitious play process $(y(t))_{t=1}^{\infty}$. Assume without loss of generality that $\max_{f\in\Delta}|U(f)|\leqslant 2^{-n}$. By (1.2), (1.3), and the multilinearity of U we get

$$U(f(t+1))-U(f(t))\geqslant\frac{1}{t+1}\sum_{i=1}^{n}(v^i(f(t))-U(f(t)))-\frac{1}{(t+1)^2}. \tag{1.6}$$

For $t\geqslant 1$ set

$$a_t=\sum_{i=1}^{n}(v^i(f(t))-U(f(t))). \tag{1.7}$$

By (1.6), because $a_t\geqslant 0$ for every $t\geqslant 1$,

$$\sum_{t=1}^{\infty}\frac{a_t}{t}<\infty. \tag{1.8}$$

We prove that

$$\lim_{T\to\infty} \frac{a_1 + a_2 + \cdots + a_T}{T} = 0. \tag{1.9}$$

Indeed, for $T \geqslant 1$ set $b_T = \sum_{t=T}^{\infty} (a_t t)$. By (1.8) $\lim_{T\to\infty} b_T = 0$. Therefore

$$\lim_{T\to\infty} \frac{b_1 + b_2 + \cdots + b_T}{T} = 0,$$

which implies that

$$\lim_{T\to\infty} \left(\frac{a_1 + a_2 + \cdots + a_T}{T} + b_{T+1} \right) = 0.$$

Since $\lim_{T\to\infty} b_{T+1} = 0$, (1.9) follows.

Let K_ε be the set of all ε-equilibrium points of Γ. By (1.9), for every $\varepsilon > 0$

$$\lim_{T\to\infty} \frac{\#\{1 \leqslant t \leqslant T : f(t) \notin K_{\varepsilon\}}}{T} = 0.$$

Therefore (1.1) is satisfied, and the proof follows from Lemma 1. ▮

Consider fixed strategy sets $Y^1, Y^2, ..., Y^n$. The game $\Gamma(u^1, u^2, ..., u^n)$, with the payoff functions $u^1, u^2, ..., u^n$, is *best response equivalent in mixed strategies* to the game $\Gamma(w^1, w^2, ..., w^n)$ if for every $i \in N$, and for every mixed strategy profile $f^{-i} \in \Delta^{-i}$, $\max_{f^i \in \Delta^i} U^i(f^i, f^{-i})$ is obtained at the same subset of Δ^i as $\max_{f^i \in \Delta^i} W^i(f^i, f^{-i})$.

A *game with identical interests* is a game which is best response equivalent in mixed strategies to a game with identical payoff functions. By Theorem A, every game with identical interests has the fictitious play property.

2. Remarks

(1) *Deriving Miyasawa's theorem.* Consider a 2-person 2×2 game described by the bimatrix $(A, B) = (a(i, j), b(i, j))_{i,j=1}^2$. We say that the game is *nondegenerate*, or that it has the *diagonal property* if $\alpha \neq 0$ and $\beta \neq 0$, where

$$\alpha = a(1, 1) - a(2, 1) - a(1, 2) + a(2, 2),$$

and

$$\beta = b(1, 1) - b(2, 1) - b(1, 2) + b(2, 2).$$

It is easily verified that every nondegenerate 2×2 game that does not have identical interests is best response equivalent in mixed strategies to a zero-sum game. So, either by Theorem A or by Robinson's theorem, every non-degenerate 2×2 game has the FPP.

(2) *Converging to a purely mixed equilibrium.* The next example shows that a fictitious play process may converge in beliefs to a purely mixed strategy profile. Consider the 2-person 2×2 pure-coordination game with $(1, 1)$ on the diagonal, and $(0, 0)$ elsewhere. If the initial belief of the players consists of a pair of pure strategies in which they do not coordinate, the fictitious play will converge to the unique purely mixed equilibrium of the game.

(3) *Utility transformations.* Deschamps [2] showed that the FPP is not necessarily invariant under increasing transformations of the utility functions that do not preserve the best response structure in mixed strategies. Monderer and Sela [8] introduced the concept of FPPS (fictitious play property in strategies). A game has the FPPS if every fictitious play process converges to a (necessarily pure strategy) equilibrium.[6] The FPPS is invariant under all increasing transformations of the utility functions. Moreover it is invariant under increasing transformations that depend on the other players' actions. That is, Player i's utility function, u^i, is transformed to v^i by

$$v^i(y) = T_{y^{-i}}(u^i(y)), \tag{2.1}$$

where $T_{y^{-i}}$ is an increasing function. The previous example shows that not every 2-person game of the form (u, u) has the FPPS. We conjecture, however, that every generic such game (and consequently every game which can be transformed to such a generic game by (2.1)) has the FPPS. Our conjecture is based on the improvement principle of [8]. According to this principle, if Player i switches to a *new* pure strategy $y^i(t+1)$, then $u^i(y^i(t+1), y^{-i}(t)) > u^i(y(t))$. Therefore if the two players never switch simultaneously, then $u(y(t))$ never decreases and it increases whenever any player switches to a new strategy. It follows that the sequence $(y(t))_{t=1}^{\infty}$ is constant after sufficiently large t. That is, it must converge. So, to prove our conjecture one can show that for a generic game of the form (u, u), simultaneous moves are impossible. We further conjecture that every 2-person game that can be transformed to a game of the form (u, u) by (2.1) has the FPP. Note, however, that the improvement principle does not hold[7] for n-person games with $n \geqslant 3$.

[6] Their definitions involves the tie-breaking rules which assumes that a player never switches to a new pure strategy if his previous action is a best response to his beliefs.

[7] However, it does hold with the other definition of fictitious play, where each player believes that all other players behave according to a fixed mixed joint strategy.

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