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Noncooperative general exchange with a continuum of traders: Two models

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Price formation and trade in a large exchange economy is modelled as a non-atomic strategic game in two contrasting forms. (1) The 'pay-later' form uses paper money or IOUs which the players must redeem at the final accounting or face overdraft penalties. (2) The 'cash-in-advance' form uses a valuable commodity as money with no need for a central clearing house. Several results connecting *strategic equilibrium* (Cournot–Nash) and *competitive equilibrium* (Walras) are obtained for (1) and (2). In the final section, a basic problem of measurability when strategies are selected independently by a continuum of agents is raised, and a way of resolving it is proposed.

Key words: Noncooperative game; Strategic equilibrium; Competitive equilibrium

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1. Introduction

Non-cooperative game models have been playing an increasing role in the study of price formation and trade in pure exchange economies.[1] In some of these models, now known as trading-post games, a specified monetary medium is used for buying and selling the other commodities at single-good trading posts – the price at each post being simply the money/good ratio at that post.[2] The money may be intrinsically valuable or it may be worthless paper, and many variations are possible in the rules governing its use.

One point of interest is the appearance of oligopoly and liquidity effects when markets are thin or when money is scarce. *Strategic* behavior then takes precedence over *competitive* behavior, and diminished trade and Pareto

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[1] Mas-Colell (1980a, b).

[2] Shapley and Shubik (1977).

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inefficiency often results.[3] The reverse phenomenon is also of interest, namely, the *disappearance* of these effects when money (or credit) is plentiful and every trader is small relative to the whole. There is an example already in Shubik's first paper (1973) in which the unique Cournot–Nash strategic equilibrium (SE) converges to a classical Walras competitive equilibrium (CE) as the number of traders of each type tends to infinity. Such examples point to a general principle of asymptotic equivalence between the two equilibrium concepts. Our aim in this paper will be to examine the essential basis of this equivalence by exploring the limiting case itself – that is, by working directly with a non-atomic continuum of traders.[4] In such a world there is of course no oligopoly. But we shall see that the smallness of the individual traders is not sufficient to wipe out the effects of illiquidity, unless further conditions on the availability of money or credit are met.

For better perspective, we shall work with two contrasting sets of rules defining the monetary instrument and its manner of use. In our first model, which is a non-atomic variant of the finite market of Shubik and Wilson (1977), the traders can create as much paper money as they please, but heavy spending is inhibited by overdraft penalties for those who are caught short when accounts are settled. A rather general equivalence between SE and CE can be established for this model when certain requirements are met concerning nonsatiation of the traders, desirability of the goods, and adequacy of the penalties.[5] (See sections 3, 4, Theorems 1–3).

Our second model is a non-atomic version of 'cash on the barrelhead' trading[6] in which the tangible, monetary good is valuable in itself as well as being the medium of exchange. In contrast with the previous 'credit' type of model, there is no worry about insolvency and no necessity for a central clearing house to balance the accounts. The market is truly decentralized. Our main result states that if money is well-liked and plentiful throughout the economy, then any SE will find 'most' of the traders behaving competitively – that is, optimizing over ordinary fixed-price Walrasian budget sets. (See sections 5–7, esp. Theorem 4.) Moreover, if there are only a finite number of *types* of traders, or if stronger conditions are imposed on the utilities, the 'most' statement can be replaced by an 'almost all' statement. (See section 8, Theorems 5 and 6.)

In an appendix we take up a general methodological, measure-theoretic question that arises when one attempts to model a strategic-form game having a continuum of players making separate decisions. The question, in brief, is how to arrange matters both technically and conceptually so that the

[3] Shapley (1976), Dubey and Rogawski (1990).

[4] Jaynes et al. (1978); cf. also Aumann (1964) and Schmeidler (1973).

[5] Somewhat similar conclusions are reached by Postlewaite and Schmeidler (1978), who consider a comparable model from the finite, asymptotic standpoint.

[6] Shapley and Shubik (1977).

integrals or other functional operators that describe the outcome-in-the-large will be well defined. (See appendix; also section 3.)

2. The non-atomic trading economy $\mathscr{E}$

Let $\{T, \mathscr{C}, \mu\}$ be a non-atomic measure space where T denotes the set of traders, $\mathscr{C}$ is the σ-algebra of measurable subsets of T, and μ is a non-atomic, non-negative, finite population measure on $\{T, \mathscr{C}\}$.[7] Trade occurs in m commodities, and we shall denote by Ω^m the nonnegative orthant of $\mathbb{R}^m$. Vectors in Ω^m may represent either commodity bundles or price vectors. For any $x \in \Omega^m$, x_j is the jth component of x. The symbol 0 denotes the origin of $\mathbb{R}^m$ and also the number zero; the meaning will be clear from the context. A set $S \in \mathscr{C}$ is *null* if $\mu(S)=0$, otherwise *non-null*. The phrases 'almost all', 'almost everywhere', or 'a.a.', 'a.e.' (referring to traders) will mean all traders except for a null set.

Given $\{T, \mathscr{C}, \mu\}$, the remaining data of an economy $\mathscr{E}$ are the traders' endowments and preferences. The endowments are written a^t, $t \in T$; we assume that the function $a: T \to \Omega^m$ is integrable and that $\int a_j^t \, \mathrm{d}u > 0$ for $j=1,\dots,m$ – in other words, every named commodity is actually present in the economy. The preferences are described by utility functions $u^t: \Omega^m \to \mathbb{R}$; we assume for each t that $u^t(x)$ is continuous, concave and nondecreasing in the m-dimensional variable x, and also that the function $u: T \times \Omega^m \to \mathbb{R}$ given by $u(t,x)=u^t(x)$ is measurable (where $T \times \Omega^m$ is equipped with the σ-field generated by the product of $\mathscr{C}$ and the Borel sets of Ω^m). If $u^t(x)$ is strictly increasing as a function of the component x_j, we shall say that t *desires* j, and if x is such that $u^t(x)$ maximizes $u^t(\cdot)$ over Ω^m we shall say that x *satiates* t.

An *allocation* is a measurable function $z: T \to \Omega^m$ with $\int_T z^t \, \mathrm{d}\mu \leq \int_T a^t \, \mathrm{d}\mu$; it describes the results of a possible redistribution of the goods, with possibly some waste. A *competitive equilibrium* (CE) is an ordered pair $\langle p, z \rangle$ where $p \in \Omega^m$ is a price vector and z is an allocation such that z^t is optimal in t's *budget set* for a.a. t. In other words, following Aumann (1964), we have a.e.

$$\begin{cases} z^t \in B^t(p), \quad \text{and} \\ u^t(z^t) = \max\,\{u^t(x): x \in B^t(p)\}, \end{cases} \tag{2.1}$$

where

$$B^t(p) = \{x \in \Omega^m: px \leq pa^t\}. \tag{2.2}$$

We shall call an allocation z [or a price p] *competitive* if there is a price p

[7]Without serious loss of generality, $\{T, \mathscr{C}, \mu\}$ may be taken as the closed unit interval with the Borel sets and Lebesgue measure.

[an allocation z] such that $\langle p,z\rangle$ is a CE. Of course, if p is a CE then so is $\langle Kp,z\rangle$, for any $K>0$, and so also is $\langle p,z'\rangle$ if z differs from z' only on a null set. We shall call a CE $\langle p,z\rangle$ *normalized* if $|p|=1$, where $|\cdot|$ denotes (say) the Euclidean norm, and call it *tight* if (2.1) holds for all $t\in T$ for which the maximum in (2.1) is attained. (If all prices in p are positive, it will be attained for all t because of the continuity of the u^t.) CEs with $|p|=0$ are not excluded, but will be called *improper* if they occur; note that impropriety implies the existence of an allocation that satiates almost all traders. Note also that for any CE $\langle p,z\rangle$ there is a z' agreeing a.e. with z such that $\langle p,z'\rangle$ is a tight CE. Thus, the tight, normalized CEs serve as representatives for the equivalence classes of proper CEs.

Let us now recall the notion of 'shadow prices of income' at a CE.[8] Suppose $\langle p,z\rangle$ is a CE. Then for a.a. t the bundle z^t maximizes the concave function $u^t(x^t)$ over the convex budget set $B^t(p)$. A *shadow price for* t is a number $\lambda^t\geq 0$ such that the same bundle z^t also maximizes the Lagrangean function

$$u^t(x^t)+\lambda^t(pa^t-px^t), \tag{2.3}$$

over the whole orthant Ω^m. If any such number exists there is a smallest one; denote it by $\lambda^t_{\langle p,z\rangle}$. If no such number exists, or if z^t does not maximize u^t over $B^t(p)$ – i.e., does not satisfy (2.1), then we formally set $\lambda^t_{\langle p,z\rangle}=\infty$. Note the inverse homogeneity of this shadow price: $\lambda^t_{\langle Kp,z\rangle}=\lambda^t_{\langle p,z\rangle}/K$ for any $K>0$.

3. First model: Trade with fiat money and credit

In order to make the exchange economy $\mathscr{E}$ into a strategic-form game $\Gamma(\mathscr{E})$ we must define the strategy choices open to the traders and specify how the market mechanism converts these choices – called collectively a *strategy selection* – into a trading outcome. For each commodity, there will be a decentralized *trading post*, to which the traders bring the commodity to be sold and the money for its purchase.

There are many ways to model the monetary instrument. In our first model it is just paper, having no value outside the marketplace. One may think of checks or IOUs written by the traders, or a trading script furnished interest-free by a bank or clearing house. A trader may spend as much as he wants of this money, but if at the end of trading he is in the red, with not enough income from his sales to cover his purchases, then he is declared 'overdrawn' and is penalized. We do not attempt to model a bankruptcy proceeding, however, with assets being seized and creditors short-changed. Instead, we pay off the creditors exogenously and deliver all goods as

[8]Henceforth shortened to 'shadow prices'.

ordered, thereby escaping the 'domino' effect when one default triggers another. We then adjust downward the *utilities* of the traders that are overdrawn, exacting (in the simplest case) penalties proportional to the overdrafts.[9] On the other hand, a positive amount of this paper money at the end of the game is assumed to be worthless to its holder.

The formal treatment begins as follows: Trader t's *strategy set* is

$$\Sigma^t = \{s^t = (q^t, r^t): q^t \in \Omega^m, r^t \in \Omega^m, q^t_j \leq a^t_j, j = 1, \ldots, m\}. \tag{3.1}$$

Here, q^t_j the quantity of good j sent to trading post j to be sold, while r^t_j is the remittance of money sent to the same post to purchase good j.[10] Of course it is likely (speaking intuitively) that either q^t_j or r^t_j will be zero – why trade with yourself? – but we shall not make this a condition in the definition of the strategies.[11]

Given a selection of strategies $s \in \Sigma = \times_{t \in T} \Sigma^t$, the next task is to determine prices. The general principle is that each trading post should clear its own books, independently of the others. In the finite-trader case, the application of this principle is straightforward: the trading post adds up the money received and the goods received and sets the price equal to the ratio of the two totals. But a peculiar difficulty arises when, as here, we have a non-atomic measure space of traders. The analogous procedure to the above would be to set $p_j = \int_T r^t_j \,\mathrm{d}\mu / \int_T q^t_j \,\mathrm{d}\mu$. The difficulty is that we are not sure that these integrals exist. Suppose a non-measurable set of traders take it into their heads to spend \$1.00 apiece, while all others spend \$2.00? What then will the price be?

Some readers will not be particularly bothered by this question. Experience tells us that when a model makes good economic sense, then concerns about mathematical pathologies seldom actually materialize; they are only ghosts, to be exorcised by suitable abjurations in a technical appendix. Perhaps that is the case here. But we are working with a new type of model whose 'good economic sense' may not be fully apparent, and the issue we are raising is as much conceptual as technical. It goes to the heart of a fundamental distinction that separates game-theoretic models from the more

[9] Such disutility may be thought of as arising from the cost of borrowing money to cover the debt or liquidating other assets not represented in the model, or the forfeit of some bond or collateral posted with the market organizers, etc. Or, less specifically, we may think of it simply as a measure of the damage done to the trader's credit rating.

[10] Alliteration aside, the term 'remittance' replaces the less accurate 'bid' found in some earlier papers. In economics as in Bridge, a 'bid' is a conditional act, requiring acceptance by another party before it becomes a contract. Our present quantities and remittances are unconditional transfers or deliveries whose acceptance is not in doubt.

[11] It seems rather unrealistic – in a model or in real life – to impose a restriction on individual agents that is easily evaded by sets of two or more, or by a single agent pretending to be several. Nevertheless, we shall include this case as a possible rule in our second model; see (5.3).

familiar behavioristic approach: Is the economic agent a decisionmaker or an automation?

The technical problem can and will be sidestepped. In the body of the paper we shall simply take our SE solutions to be measurable selections *by definition.*[12] In effect, we merge the rules of the solution with the rules of the game, producing thereby a model that is no longer a game, but a behavioral process. But this expedient cannot be considered fully satisfactory. Substantial investigations in the area of rational decisionmaking ought not to be based on an incomplete game structure, in which a large class of possible strategy selections fail to yield a well-defined outcome.

Accordingly, in the appendix we shall outline a novel way of attacking the conceptual difficulty, using a form of collective *strategies for small coalitions* to bridge the modelling gap between independent decisionmaking by individuals and the measurable (and indeed additive) behavior of collectivities. By our preliminary treatment we wish to draw attention to the problem in a way that may stimulate further thinking on the subject.

The mechanism of trade. We now assume that the traders have made a measurable selection, $s=(q,r)\in\Sigma$, and describe the consequences. First each trading post j establishes its price:[13]

$$p_j=\begin{cases}\int r_j/\int q_j & \text{if } \int q_j>0,\\ \infty & \text{if } \int q_j=0 \text{ and } \int r_j>0,\\ \text{undefined} & \text{if } \int q_j=\int r_j=0;\end{cases} \tag{3.2}$$

then executes all the buy and sell orders at this price, and finally delivers to each trader t the proceeds, namely, either the amount of goods purchased,

$$\begin{cases}r_j^t/p_j & \text{if } 0<p_j<\infty,\\ 1 & \text{if } p_j=0 \text{ and } r_j^t>0,\\ 0 & \text{otherwise,}\end{cases} \tag{3.3}$$

or the amount of money realized from the goods sold,

[12] Cf. Schmeidler (1973). It would be better, of course, to be able to derive the measurability of the SEs as a theorem; our remark at the end of the appendix suggests a way of achieving this in some circumstances. Analogously in the cooperative theory, it can sometimes be proved that all core allocations are countably additive, though a priori they are only finitely additive; see Kannai (1969), Rosenmüller (1971), Schmeidler (1973).

[13] Beginning here, we shall use the short form '$\int r_j$' for '$\int_T r_j^t \,d\mu(t)$', etc.

$$\begin{cases} p_j q_j^t & \text{if} \quad 0 < p_j < \infty, \\ 1 & \text{if} \quad p_j = \infty \text{ and } q_j^t > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

– or both. This will clear the market at the trading posts where a positive, finite price is formed [top line of (3.3) and (3.4)]. If $p_j = 0$ or $p_j = \infty$ we call the trading post *lopsided*, and if both the goods and the money sent in have measure zero, so that p_j is undefined, we call the post *inactive*. These situations will be discussed more fully in section 4, including an explanation of the '1' that strangely appears in the second lines of (3.3) or (3.4).

When the buying and selling is finished, the final holdings of goods and money are described by the function $\tilde{z} = (z, \tilde{z}_{m+1})$: $T \to \Omega^m \times \mathbb{R}$, where

$$\begin{cases} z_j^t & = a_j^t - q_j^t + r_j^t / p_j, \quad j = 1, \ldots, m, \\ \tilde{z}_{m+1}^t & = -\sum_{j=1}^{m} r_j^t + \sum_{j=1}^{m} p_j q_j^t, \end{cases} \tag{3.5}$$

with whatever adjustments may be required for lopsided or inactive trading posts according to (3.3) and (3.4).

For overdrafts, we shall adopt for simplicity[14] a linear separable penalty ζ: $T \to \Omega$, measurable in t, making the overall utility to t of his final holding $\tilde{z}^t$ equal to

$$U^t(\tilde{z}^t) = u^t(z^t) + \zeta^t \min\{0, \tilde{z}_{m+1}^t\}. \tag{3.6}$$

The game in its strategic form is therefore summed up in the payoff function

$$\Pi^t(s) = U^t(\tilde{z}^t), \quad t \in T, \tag{3.7}$$

in which $\tilde{z} = \tilde{z}(s)$ is linked to s through (3.2)–(3.5) while s ranges over the measurable elements of Σ. We shall denote this game in strategic form by $\Gamma_\zeta(\mathscr{E})$ to remind ourselves that it depends on the penalty coefficients ζ^t as well as on the data of $\mathscr{E}$.

Strategic equilibrium. A *strategic equilibrium* (*SE*) of $\Gamma_\zeta(\mathscr{E})$ is defined to be a measurable[15] selection $s_{\#} \in \Sigma$ such that, for a.a. t,

[14]More general penalties are discussed in section 4.

[15]It may be noted that if the word 'measurable' were dropped, the present measurable SEs would remain SEs no matter how one might extend the $\Pi^t(s)$ to more general selections. This is because varying the decision of one individual does not affect measurability. But additional, nommeasurable SEs might also appear. (See the appendix).

$$\Pi^t(s_\#)=\max\{\Pi^t(s_\# \mid s^t): s^t\in\Sigma^t\}, \tag{3.8}$$

where the notation '$s_\# \mid s^t$' means the selection obtained by replacing $s^t{}_\#$ in $s_\#$ by s^t. By an *SE allocation* of $\Gamma_\zeta(\mathscr{E})$ we shall mean a final allocation of z (goods only) that results from some SE of $\Gamma_\zeta(\mathscr{E})$.

Note the homogeneity: if (q,r) is an SE of $\Gamma_\zeta(\mathscr{E})$, then (q,Kr) is an SE of $\Gamma_{\zeta/K}(\mathscr{E})$ for any $K>0$.

There is always the class of trivial SEs in which $(q^t,r^t)=(0,0)$ for a.a. t, rendering all trading posts inactive. At other SEs some posts may be inactive, some not. This is not unrealistic. Potentially profitable markets often fail to form in practice simply because not enought people are willing to make the initial commitment. But we shall be interested primarily in *active* SEs, namely, those in which all $\int q_j$ and $\int r_j$ are positive.[16]

Competitive vs. strategic equilibrium. The relationship in this model between the CE and SE theories is addressed in the following three theorems. They present conditions under which, respectively, all SE allocations are CE, all CE allocations are SE, and the sets of SEs and CEs coincide.

Theorem 1. Every active SE allocation of $\Gamma_\zeta(\mathscr{E})$ that leaves almost every trader unsatiated is competitive for $\mathscr{E}$.

Theorem 2. If each good is desired by a non-null set of traders, then each competitive allocation z of $\mathscr{E}$ is an active SE allocation of $\Gamma_\zeta(\mathscr{E})$ provided that $K\zeta^t\geq\lambda^t\langle p,z\rangle$ a.e. for some $K>0$ and some p such that $\langle p,z\rangle$ is a CE of $\mathscr{E}$.

Theorem 3 requires a further definition, which identifies the highest 'marginal utility of income' at normalized prices that each player enjoys in a CE.[17] Define, for each $t\in T$,

$$\hat\lambda^t=\hat\lambda^t[\mathscr{E}]=\sup\{|p|\lambda^t\langle p,z\rangle: \langle p,z\rangle \text{ is a tight CE of } \mathscr{E}\}; \tag{3.9}$$

this may possibly be infinite. ['Tightness' here serves to exclude CEs in which t belongs to the null set of traders that violate (2.1).] If $\mathscr{E}$ has no CEs, then we interpret (3.9) to define $\hat\lambda^t[\mathscr{E}]=0$.

Theorem 3. Let each good be desired by a non-null set of traders, and let each allocation satiate at most a null set of traders. Then the set of CE allocations of $\mathscr{E}$ and the set of active SE allocations of $\Gamma_\zeta(\mathscr{E})$ coincide, provided that $K\zeta^t\geq\hat\lambda^t[\mathscr{E}]$ a.e. for some positive constant K.

[16]Cf. our definition of 'open' SEs in section 5.

[17]See the next-to-last remark in section 4.

Proof of Theorem 1. Let p and $\tilde{z}$ be the price and final outcome at an active SE. It is directly verified that $\int z = \int a$ and that $\int \tilde{z}_{m+1} = 0$. We claim that $\tilde{z}_{m+1} = 0$ a.e. Indeed, if this were not true, then $\tilde{z}_{m+1}$ would be positive on a non-null set $S \in \mathscr{C}$. Since almost all players are unsatiated at z by hypothesis, we may assume that all the members of S are unsatiated. But they still have spending money, so that could all improve their individual payoffs, contrary to the definition of SE. Hence $\tilde{z}_{m+1} = 0$ a.e. It is now immediate that $\langle p, z \rangle$ is a CE. □

Proof of Theorem 2. Let p, z, ζ, and K satisfy the hypotheses of the theorem. By the desirability assumption, $p > 0$. Define a strategy selection $s_* = (q, r,) \in \Sigma$ by

$$\begin{cases} q^t_j = \max\,[a^t_j - z^t_j, 0], \\ r^t_j = \max\,[Kp_j(z^t_j - a^t_j), 0]. \end{cases} \tag{3.10}$$

If none of the trading posts are inactive, then (3.2) and (3.3) will yield Kp as the price vector of $\Gamma_\zeta(\mathscr{E})$ and furthermore (since $0 < p < \infty$ and there is no waste) will yield $\tilde{z}$ with $\tilde{z}_{m+1} = 0$ as the final allocation. Is s_* an SE?

By (2.3) we have

$$u^t(z^t) = \max\,\{u^t(x) + \lambda^t_{\langle p,z \rangle} p(a^t - x) \colon x \in \Omega^m\} \tag{3.11}$$

for almost all traders t. For such a trader the payoff is

$$\Pi(s_*) = U^t(\tilde{z}^t) = u^t(z^t),$$

there being no penalty in view of $p(a^t - z^t) = 0$. If t could improve, a bundle $\tilde{x}^t$ would exist such that

$$U^t(\tilde{x}^t) = u^t(x^t) + \zeta^t \min\,[0, \tilde{x}^t_{m+1}] > u^t(z^t).$$

But, by (3.11),

$$u^t(x^t) + \lambda^t_{\langle p,z \rangle} p(a^t - x^t) \leq u^t(z^t),$$

and hence

$$\lambda^t_{\langle p,z \rangle} p(a^t - x^t) < \zeta^t \min\,[0, \tilde{x}^t_{m+1}].$$

Since $Kp(a^t - x^t) = \tilde{x}^t_{m+1}$, this would imply that

$$\lambda^t_{\langle p,z \rangle} \tilde{x}^t_{m+1} < K\zeta^t \min\,[0, \tilde{x}^t_{m+1}].$$

This statement is clearly false of $\tilde{x}^t_{m+1} \geq 0$, whereas $\tilde{x}^t_{m+1} < 0$ it implies that

$K\zeta^t < \lambda^t_{\langle p,z \rangle}$. By hypothesis, this can happen only on a null set. So the set of players who can improve is null, and s_* is an SE.

What if (3.10) yields one or more inactive trading posts? This would mean that the original CE just happened to call for zero trade in one or more goods i.e., $z^t_j = a^t_j$ a.e. Fortunately, this sort of inactivity is a 'removable singularity', since there is an equilibrium price p_j available for any such j, despite the absence of trade.[18] Since $\int a_j > 0$, there is an $\varepsilon_j > 0$ and a non-null $S_j \in \mathscr{C}$ such that $a^t_j \geq \varepsilon_j$ for all $t \in S_j$. Let us modify the selection s_* defined above so that $q^t_j = \varepsilon_j$ and $r^t_j = p_j \varepsilon_j$ for all $t \in S_j$, and do the same for any other trading posts that happen to be inactive. After these modifications, the selection is obviously still feasible (and measurable), and since the discontinuities in (3.3) are avoided there is now no waste and it yields the desired final allocation z. The rest of the proof is as before. □

Proof of Theorem 3. If z is an active SE allocation of $\Gamma_\zeta(\mathscr{E})$, then by Theorem 1 it is a CE allocation of $\mathscr{E}$.

Going the other way, let $\langle p,z \rangle$ be a CE of $\mathscr{E}$. Let T' be the set of 'well-behaved' traders in T for whom (2.1) holds. Of course, $\mu(T') = \mu(T)$. Moreover, we can construct a tight CE with price vector p and allocation z' that agrees with z on T'. It follows that $\lambda^t_{\langle p,z' \rangle} = \lambda^t_{\langle p,z \rangle}$ a.e. We therefore have

$$K\zeta^t \geq \hat{\lambda}^t[\mathscr{E}] \geq |p| \lambda^t_{\langle p,z' \rangle} = |p| \lambda^t_{\langle p,z \rangle}$$

almost everywhere. Taking the constant in Theorem 2 to be $K/|p|$, we conclude that z is an active SE allocation. □

4. Example and remarks

Theorems 1–3 may be better understood if we deliberately violate their conditions. In the following example, the 'equivalence' between CE and SE breaks down in both directions.

We take $\{T, \mathscr{C}, \mu\}$ to be the unit interval with the Borel sets and Lebesgue measure. Let $m=1$ and let $a^t_1 = 3$ for $t \in [0, \frac{1}{2}]$ (the 'rich' traders), and $a^t_1 = 1$ for $t \in (\frac{1}{2}, 1]$ (the 'poor' traders). Let them all have the same basic utility function,

$$u^t(z) \equiv u(x) = \begin{cases} 4x_1 - x_1^2 & \text{for } 0 \leq x_1 \leq 2, \\ 4 & \text{for } x_1 > 2, \end{cases}$$

and the same overdraft disincentives,

$$U^t(\tilde{x}) = u^t(x) + L \min\{0, \tilde{x}_2\},$$

[18] Called *virtual price* in Shapley (1976); cf. also the discussion in section 5 below.

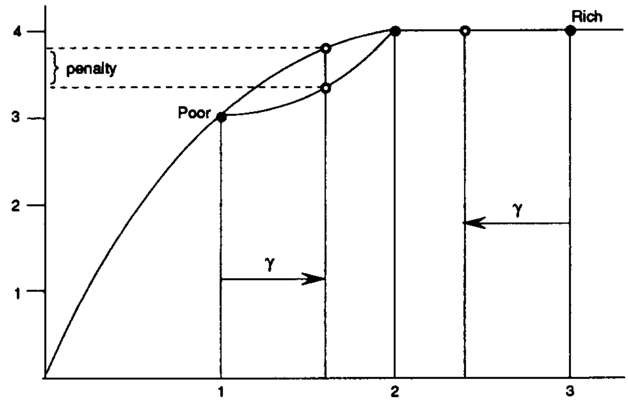

Fig. 1

where L is some positive constant (see fig. 1). We first describe a parametric family of SEs. Let $0 \leq \gamma < 1$, and let σ_γ denote the following strategy selection:

$$\text{rich } t\colon \begin{cases} q^t = \gamma \\ r^t = 0 \end{cases}; \qquad \text{poor } t\colon \begin{cases} q^t = 0 \\ r^t = \gamma(1-\gamma)/2L \end{cases}.$$

The resulting price, if $\gamma \neq 0$, is

$$p = p_1 = (1-\gamma)/2L,$$

and the final holdings, for any γ, are

$$\begin{cases} \text{rich } t\colon z^t = (3-\gamma, \gamma(1-\gamma)/2L), \\ \text{poor } t\colon z^t = (1+\gamma, -\gamma(1-\gamma)/2L), \end{cases}$$

yielding utilities of 4 and $3+\gamma^2$, respectively; the latter is indicated in fig. 1 by the lower curved segment.

Thus, at the end of the game the 'rich' are still satiatied and have picked up some worthless money. The 'poor' are overdrawn, but have $1+\gamma$ units of the good to enjoy. Their marginal utility at that level of consumption works out to be 2 $(1-\gamma)$, which by no accident is equal to pL – the penalty rate per unit of the good. So they, like their richer brethren, are making best responses to the overall strategy selection, and σ_γ is an SE as claimed.

Note that σ_γ continues to be an SE at $\gamma = 0$, but is inactive. On the other hand, at $\gamma = 1$ the trading post is lopsided (see below) and is not in equilibrium.

Although these strategy selections σ_γ are not the only SEs, they suffice to show that the SE *allocations* include all symmetric transfers from 'rich' to 'poor', up to the point that equalizes their holdings. But it is easily shown that there is no SE that yields full equality between the types – i.e., $z^t \equiv 2$.

The situation with respect to CEs is very simple since there is only one good. In fact, there are just two tight CE allocations, namely $z^t \equiv a^t$ and $z^t \equiv 2$. The first is supported by any $p > 0$, since the 'rich' have no desire to sell and the 'poor' can't afford to buy. This group of CEs is more or less equivalent to the inactive SE σ_0, with its indeterminate price. The second is supported only by the improper price $p = 0$: here the poor can glut themselves and the righ don't care. It corresponds more or less to the selection σ_1, which, as already remarked, is a limit of SEs but not an SE itself.

To sum up, this example exhibits *satiation*, which makes overdrafts possible, and *lack of desirability*, which makes improper CEs possible. As a result, the sets of SE and CE allocations are quite different.

Remark on lopsided trading posts. It is not surprising that there should be a singularity in the rules when a trading post receives goods but no money, or money but no goods; the only question is what to do about it. The finite model offers some guidance. There it made sense for an individual trader, assuming that no one else is going to attempt to buy one of the goods, to move in and make a 'killing' by buying out the entire stock at the trading post at an arbitrarily low price. Our present, nonatomic trader t, however, lives in a world of infinitesimals. His number r_j^t is a density (more precisely, an integrand), not comparable to a globally significant quantity like $\int q_j$. In order to clear out the trading post's entire stock, our lone trader would have to achieve not a density equal to the number $\int q_j$, but an infinite density which would make him an atom in the distribution of good j. This would not only require an extension of our mathematical model, but would also create new problems,[19] and the added complications would not be worth the trouble considering that such lopsided situations are generally far from the equilibria we are interested in.

For that reason we have adopted the admittedly artificial device of awarding the nominal amount of 1 density-unit of good j or money to the buyers or sellers of j when they fail to form a set of positive measure.[20] This simple expedient serves to prevent the appearance of spurious SEs in these extreme situations and allows us to concentrate on more interesting questions.

Remark on the proof of Theorem 2. The selection s_* at (3.10) is the most direct route to the desired allocation z, since no one both buys and sells at

[19]For example, how does one divide $\int q_j$ when the set of buyers is null but uncountable?
[20]Any other positive number would do as well as 1.

the same trading post. The volume of trade in each good is therefore minimized. But the proof might equally well have used other selections yielding the same price and income. For example, the selection $s_{**}=(q,r)$ with

$$\begin{Bmatrix} q_j^t=a_j^t \\ r_j^t=Kp_j z_j^t \end{Bmatrix} \quad \text{all } j=1,\dots,m, \text{ all } t\in T, \tag{4.1}$$

would avoid the need for special treatment of inactive trading posts. In particular, (4.1) shows that Theorem 2 remains valid if the 'sell-all' condition $q_j^t \equiv a_j^t$ is imposed, requiring all goods to pass through the market before being consumed. Such a condition will be imposed on one of the variants in section 5.

Remark on the rate of penalty. The free factor of K in Theorems 2 and 3, which adds greatly to the strength of the conclusions, comes from the fact that the model as a whole, except for the overdraft penalty, is independent of the size of the money unit. Multiplying all remittances by K has only one 'real' effect, namely, to multiply by K the disutility of overspending. If K is large, the SEs are in a certain sense less likely to involve overdrafts, and therefore more likely to yield competitive allocations.[21]

This feature of the model might indeed be considered contrary to the spirit of the penalty rule. If so, it can easily be corrected by 'indexing' the penalties, simply replacing (3.6) by

$$U^t(\tilde{z}^t)=u^t(x^t)+\frac{\zeta^t}{|p|}\min\{0,\tilde{z}^t_{m+1}\}, \quad \text{all} \quad t\in T. \tag{4.2}$$

The entire model is now unaffected by price inflation or deflation, and Theorem 2 takes the following new form:

Theorem 2′. If each good is desired by a non-null set of traders, and if $\langle p,z\rangle$ is a normalized CE of $\mathscr{E}$, then z is an active SE allocation of the 'indexed' game $\hat{\Gamma}_\zeta(\mathscr{E})$, provided that $\zeta^t \geq \lambda^t_{\langle p,z\rangle}$ a.e.

The proof is essentially the same. A similar adjustment can be made to Theorem 3.

Remark on more general penalties. The linear separable form of the overdraft penalty in (3.6) is not at all essential. It would be sufficient to require in $\Gamma_\zeta(\mathscr{E})$ that

[21]It is easy to see that if (q,r) is an SE without overdrafts, then so is (q,Kr) for all $K>1$.

$$\begin{cases} U^t(\tilde{z}^t) = u^t(z^t) & \text{if } \tilde{z}^t_{m+1} \geq 0, \\ U^t(\tilde{z}^t) \leq u^t(z^t) + \zeta^t \tilde{z}^t_{m+1}, & \text{if } \tilde{z}^t_{m+1} < 0. \end{cases} \tag{4.3}$$

In other words, the linear penalty may be replaced by anything that is harsher, including penalties that depend on the holding of other goods. Concavity does not matter, nor does continuity or monotonicity, but U^t must be taken measurable across $t \in T$.

Remark on the bound $\hat{\lambda}[\mathscr{E}]$ in Theorem 3. If each u^t is smooth and strictly concave, it is well known that in the finite-trader case, there are generically at most a finite number of (normalized) CEs.[22] From this it follows immediately that the bound $\hat{\lambda}[\mathscr{E}]$ in the finite-type case (with types of equal measure; see section 8) will be finite a.e., and we can at least assert the *existence* of a set of penalty rates that are sufficient for SE/CE equivalence, even though the determination of $\hat{\lambda}[\mathscr{E}]$ from (3.9) might be extremely difficult.

Even without the finite-equal-type restriction, $\hat{\lambda}[\mathscr{E}]$ will exist whenever there are only finitely many normalized, tight CEs. If we already know that there are only finitely many normalized CE prices, this would be assured, for example, if the prices are all positive and the utilities are strictly concave.[23] No doubt there are milder conditions that guarantee the existence of the bound $\hat{\lambda}[\mathscr{E}]$, not involving finiteness of the set of normalized, tight CEs, but we have not explored this question.

Remark on the existence of SEs. From Theorem 2 (or 2′ or 3), we can infer the existence of SEs from the existence of CEs. Indeed, let $\mathscr{E}$ satisfy the standard assumptions which guarantee a competitive equilibrium [Aumann (1966)]. If $\langle p, z\rangle$ is a CE, then there exists a measurable λ such that each λ^t is a shadow price for t. (The measurability of λ follows from the fact that u and z are measurable.) Then Theorem 2 immediately gives the class of measurable ζ for which the game $\Gamma_\zeta(\mathscr{E})$ has SEs; e.g., $\zeta = K\lambda + \zeta'$, for any positive scalar K and any nonnegative measurable ζ'.

5. The second model: Trade using a commodity money

The rules of trade in our second model differ in several respects from the previous case. Rather than paper money, of no intrinsic worth, the traders use one of the real commodities as a medium of exchange. The terms are 'cash in advance' so there is no need for a clearing house (indeed, the market

[22]Debreu (1970).

[23]For the generic finiteness of normalized CE prices in non-atomic economies with smooth utilities see Dierker (1975).

is now completely decentralized), and no one is ever overdrawn so there is no need for penalty functions. But with the traders' spending limited to 'cash' on hand, there is the more interesting possibility that an insufficient or maldistributed money supply could affect the SEs and their relationship to the CEs. Our results in this and the next three sections will address this possibility.

The notation requires only a slight modification from that of the first model (see section 3). The economy $\mathscr{E}=\{(T,\mathscr{C},\mu)a,u\}$ is as before, except that '$m+1$' now stands for a tangible commodity and the vectors a^t, z^t, etc. now live in Ω^{m+1}, not Ω^m. The *utility functions* u^t therefore map Ω^{m+1} into $\mathbb{R}$, and from this point forward it will suffice to assume that they are merely *quasi-concave*. Our results, however, will require certain additional assumptions relating to the desirability of money, which we shall discuss in sections 6–8.

We shall simultaneously treat three different games based on the same economy $\mathscr{E}$, denoting them $\Gamma_1(\mathscr{E})$, $\Gamma_2(\mathscr{E})$, and $\Gamma_3(\mathscr{E})$.[24] They differ only in their strategy spaces Σ_i^t, $i=1,2,3$. In the *buy-and-sell* case the traders have the most latitude:

$$\Sigma_1^t=\left\{s^t=(q^t,r^t)\in\Omega^{2m}\colon q_j^t\leq a_j^t,\ j=1,\dots,m,\text{ and }\sum_{j=1}^m r_j^t\leq a_{m+1}^t\right\}. \tag{5.1}$$

In the *sell-all* case the q_j^t are predetermined:

$$\Sigma_2^t=\{s^t\in\Sigma_1^t\colon q_j^t=a_j^t,\ j=1,\dots,m\}. \tag{5.2}$$

Finally, in the *buy-or-sell* case the traders must decide between buying and selling:

$$\Sigma_3^t=\{s^t\in\Sigma_1^t\colon q_j^t r_j^t=0,\ j=1,\dots,m\}. \tag{5.3}$$

In all three variants the rules of pricing and distribution are defined exactly as in (3.2)–(3.5) with the exception of the bottom line of (3.5), which must be amended as follows:

$$z_{m+1}^t=a_{m+1}^t-\sum_{j=1}^m r_j^t+\sum_{j=1}^m p_j q_j^t. \tag{5.4}$$

The payoff function in all variants is

$$\Pi^t(s)=u^t(z^t),\quad t\in T, \tag{5.5}$$

and, as before, an SE of $\Gamma_i(\mathscr{E})$ is any measurable selection $s_{\#}\in\Sigma_i$ such that

[24]The '$(\mathscr{E})$' may be omitted when the context is clear.

$$\Pi^t(s_{\#}) = \max\{\Pi^t(s_{\#} \mid s^t): s^t \in \Sigma_i^t\}. \tag{5.6}$$

It is obvious that if s is an SE of Γ_1 that happens to belong to Σ_2 or Σ_3 then it is an SE of Γ_2 or Γ_3 as well. We shall presently see that SEs of Γ_1 that do not belong to Σ_3 are also closely related to the SEs of Γ_3.[25] The SEs of the sell-all model Γ_2 may be quite different, however.

Inactive posts and open SEs. The key differences between the present games and the fiat-money game of section 3 are the absence of overdraft penalties and the presence of spending limits. We no longer have to worry about insolvency, but there are other complications. In particular, we shall have to consider more carefully the use of 'wash sales' (as used in the proof of Theorem 2; see the discussion in section 4), and the situation at inactive trading posts.

Recall that an inactive trading post is like a 'black hole', into which an agent's goods or money may disappear without a trace. Such a post is in effect closed to trade, and no price is formed. A kind of forced equilibrium is created. Indeed, if we arbitrarily inactivate one or more trading posts, eliminating their terms from the strategy vectors of Γ_1 or Γ_3, then any SE of this modified game is automatically an SE of the original game. In particular, the null selection (i.e., no trade in any good) is always an SE in the buy-and-sell and buy-or-sell cases.

There is a less arbitrary kind of inactivity, however, that occurs when a *virtual price* can be defined for the inactive good.[26] This is a kind of local, 'spot price' for a good that is not being traded because its initial distribution just happens to be in equilibrium w.r.t. the given strategy selection and the other virtual prices, if any. Imagine a well-stocked, fixed-price Walrasian store which is 'open for business', but has no customers.

To formalize this idea, let $s=(q,r)$ be an SE of $\Gamma_i(\mathscr{E})$, and let $I(s)$ denote the set of all trading posts that are inactive at s, in the sense that

$$I(s) = \{j \in \{1,\dots,m\}: \textstyle\int q_j = \int r_j = 0\}.$$

Then s is an *open SE* of $\Gamma_i(\mathscr{E})$,[27] if we can associate a price p_j (not necessarily finite) to every $j \in I(s)$ in such a way that (q^t, r^t) maximizes $u^t(z)$ over Σ_i^t for almost all traders t, where as usual[28]

$$\begin{cases} z_j^t = a_j^t - q_j^t + r_j^t/p_j, \quad j = 1,\dots,m, \\ z_{m+1}^t = a_{m+1}^t - \sum_{j=1}^m r_j^t + \sum_{j=1}^m p_j q_j^t. \end{cases}$$

[25] See the remark at the end of this section.

[26] See Shapley (1976, p. 171).

[27] The reader will observe that the case $i=2$ is trivial: all SEs are open since no trading posts are inactive.

[28] If any p_j is 0 or ∞, these formulas must be modified according to (3.3) and (3.4).

Thus all active SEs are open SEs, but many others may be open. Even the identically-zero SE is open when the initial allocation is Pareto optimal.

Interior and competitive traders. Let $s=(q,r)$ be an open SE for $\Gamma_i(\mathscr{E})$, $i=1,2,3$, with a price vector p and a final allocation z. Let $\tilde{p}$ be defined by

$$\tilde{p}_j = p_j \text{ for } j=1,\dots,m, \quad \text{and} \quad \tilde{p}_{m+1}=1.$$

We shall call a trader t *interior* if he refrains from spending all his money:

$$\sum_{j=1}^{m} r_j^t < a_{m+1}^t, \tag{5.7}$$

and if no prices are infinite, we shall call him *competitive* if he weakly prefers z^t to every other bundle in his budget set:

$$B^t(\tilde{p}) = \{x \in \Omega^{m+1} : \tilde{p}x \leq \tilde{p}a^t\}. \tag{5.8}$$

Lemma 1. Assume that almost all traders have strictly monotonic utility for the money commodity. At any open SE of $\Gamma_i(\mathscr{E})$, $i=1,2,3$:

(i) *all prices are finite; and*
(ii) *almost all interior traders are competitive.*

Proof. (i) Let $s=(q,r)$ be an open SE of $\Gamma_i(\mathscr{E})$, and suppose $p_j=\infty$ for some j. Then, by (3.2), $\int q_j=0$ and $\int r_j>0$, so post j is neither active nor inactive, but lopsided. In particular, there is a non-null set $S\subset T$ with $r_j^t>0$, all $t\in S$, and each $t\in S$ could reduce his remittance r_j^t (but keeping it positive), and end up with more money and the same amount of the other goods. Since money is almost always desirable by assumption, almost every t's payoff would increase – a contradiction.
(ii) Since individual traders cannot affect prices, choosing $s^t\in\Sigma_i^t$ to maximize $\Pi^t(s)$ [see (5.6)] is equivalent to choosing $z^t\in\Omega^{m+1}$ to maximize $u^t(z^t)$ subject to two constraints: the spending limitation (5.1) and the money-balance equation (5.4). As u^t is quasi-concave by assumption, the former is inoperative for an interior trader, while the latter reduces to

$$z_{m+1}^t = a_{m+1}^t - \sum_{j=1}^{m} p_j z_j^t + \sum_{j=1}^{m} p_j a_j^t,$$

which is valid even if some of the p_j's are 0. Collecting terms, this may be re-stated.

$$\tilde{p}z^t = \tilde{p}a^t.$$

The conclusion is now apparent. □

Remark on open and active SEs. It often turns out (see the examples in section 6) that many traders t at an open SE are *not* interior, i.e., $\sum_{j=1}^{m} r_j^t = a_{m+1}^t$. In this case there are typically bundles in $B^t(\tilde{p})$ that t strictly prefers to the bundle he gets at the SE. Moreover, $r_j^t q_j^t = 0$ for all j for such t since t is at his spending limit and cannot afford the luxury of 'wash sales'. Thus, for a non-interior t, the difference between Σ_1^t and Σ_3^t is not serious: the best responses in Σ_1^t are available in Σ_3^t anyway. Even when t is interior, it is clear that he can obtain exactly the same set of final bundles through Σ_3^t as he can through Σ_1^t, without disturbing the rest of the traders. So the prices and allocations achieved at open SEs in Γ_1 are also achieved at open SEs in Γ_3. Indeed, the converse being obvious, the two sets coincide.

Formally, let (q,r) be an open SE of Γ_1 with prices p, and define (q',r') in Σ_3 by

$$q_j'^t = \max\left\{0, -\frac{r_j^t}{p_j} + q_j^t\right\}, \quad r_j'^t = \max\left\{0, p_j\left(-q_j^t + \frac{r_j^t}{p_j}\right)\right\}, \qquad \text{all } t \in T.$$

Clearly (q',r') is an open SE of Γ_3 with the same prices and allocations as (q,r). But, in the above transition, some trading posts that were active in (q,r) may have been rendered inactive in (q',r') with their prices changed from 'real' to 'virtual'.

6. Convergence to competitive behavior

If we are to establish a link between CEs and SEs when the latter are based on a commodity money in fixed supply, we must find conditions that will neutralize the spending limit in the SE model – a constraint that has no counterpart in the CE model. It comes down to a question of liquidity: the traders must be given enough money. But how to define 'enough'? If an SE is to have prices and allocations that are competitive (or approximately competitive in a sense to be described), then all or most of the traders must *willingly* stop spending before their limits are reached. It is easy enough to pump more and more money into the initial endowments (see below), but we shall also need a condition to assure that the money commodity holds its value vis-à-vis the other commodities as the supply increases.[29]

[29]The utility imputed to the commodity money may be regarded as coming from its economic value in consumption or production, or from its strategic value as a medium of future exchange – or a combination of the two (e.g., gold). In the present static setting it doesn't much matter which interpretation is adopted.

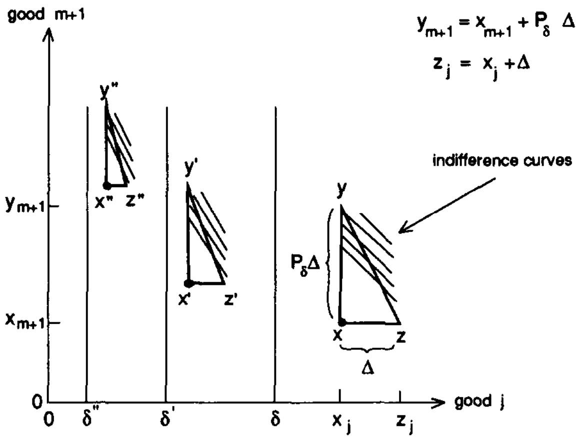

Fig. 2. Illustrating Assumption A.

It would be sufficient for us to assume a uniform upper bound on the ratios of marginal utility:

$$R_j^t(x) = \frac{\partial u^t}{\partial x_j} \bigg/ \frac{\partial u^t}{\partial x_{m+1}}, \quad j = 1, \ldots, m.$$

But, seeking a sharper result we assume less: we permit $R_j^t(x)$ to grow without bound as $x_j \to 0$ and we also allow for the possible nondifferentiability of the u^t. In the following, e_j will denote the jth unit vector of Ω^{m+1}.

Assumption A. For each $\delta > 0$ there is a measurable function P_δ: $T \to \mathbb{R}_+$ such that, for each $j = 1, \ldots, m$,

$$u^t(x + P_\delta^t \Delta e_{m+1}) > u^t(x + \Delta e_j) \tag{6.1}$$

holds a.e. for all sufficiently small $\Delta > 0$ and all $x \in \Omega^{m+1}$ with $x_j \geq \delta$.

A somewhat stronger version will sometimes be useful:

Assumption A′. As in Assumption A, but for each δ, P_δ is 'essentially bounded' in the sense that there is a constant $\bar{P}_\delta$ such that $P_\delta^t \leq \bar{P}_\delta$ a.e.

In either version, the function P_δ serves to keep the relative value of money w.r.t. each of the other goods j away from zero, measurably in $t \in T$ and uniformly in x_l, $l \neq j$. Thus, if the utilities are quasi-concave and differentiable, and if the above-mentioned upper bounds do exist, say $R_j^t(x) \leq \bar{R}_j^t$, then we could simply take $P_\delta^t = 1/\max_j \bar{R}_j^t$ for all $\delta > 0$. (See fig. 2).

Under this assumption, in either version, we shall show that if money is 'everywhere plentiful' amd 'well distributed', then the SE allocations will be 'approximately competitive'. Of course, the terms in quotation marks will have to be given precise meanings.

Since we wish to study the role of the initial distribution of money, it is convenient to introduce the notation $\mathscr{E}=(\bar{\mathscr{E}}, a_{m+1})$ where $\bar{\mathscr{E}}$ represents all the data of $\mathscr{E}$ except for the function a_{m+1}: $T\to\Omega$. In this section and the two following we focus on the effect of increasing the money supply attached to a fixed $\bar{\mathscr{E}}$. Accordingly, we set up a sequence of economies

$$\mathscr{E}^{\nu}=(\bar{\mathscr{E}}, a_{m+1}^{\nu}), \quad \nu=1,2,\ldots,^{30} \tag{6.2}$$

and assume that

$$\alpha_0^{\nu}\to\infty \quad \text{as} \quad \nu\to\infty, \tag{6.3}$$

where α_0^{ν} denotes the *essential infimum* of the set $\{a_{m+1}^{t\nu}: t\in T\}$ – i.e., the largest number α such that the set of t with $a_{m+1}^{t\nu}<\alpha$ has measure zero.[31]

Suppose $\Gamma_i(\mathscr{E}^{\nu})$, $i=1,2$ or 3, possesses at least one open SE for each ν, say $s_i^{\nu}=(q^{\nu}, r^{\nu})_i$. Recall that $t\in T$ is *interior* at s_i^{ν} if and only if $\sum_{j=1}^{m} r_j^{t\nu}<a_{m+1}^{t\nu}$, and let Q_i^{ν} denote the set of traders in $\Gamma_i(\mathscr{E}^{\nu})$ who are *not* interior at s_i^{ν}. Thus,

$$Q_i^{\nu}=\left\{t\in T: \sum_{j=1}^{m} r_j^{t\nu}=a_{m+1}^{t\nu}\right\}.$$

Armed with Lemma 1, which tells us that at an open SE the set of traders who are interior but not competitive has measure zero, we claim that $\mu(Q_i^{\nu})$ is a reasonable measure of the 'noncompetitiveness' of that SE – i.e., of how far it falls short of being a CE.

We now state the main theorem.

Theorem 4. Let the sequence $\mathscr{E}^{\nu}=(\mathscr{E}, a_{m+1}^{\nu})$, $\nu=1,2,\ldots,$ satisfy (6.3), and let the u^t satisy Assumption A with the bounding function P_δ. Then, for $i=1,2,3$ we have:

(i) *If the ratio*[32] $\int a_{m+1}^{\nu}/\alpha_0^{\nu}$ *is bounded as* $\nu\to\infty$, *then* $\mu(Q_i^{\nu})\to 0$.

[30]Notation: We shall consistently put the tag 'ν' in the superscript position, where it may collide with the 't' that owns that spot. When both letters occur, 'ν' will be given the outside position. The divergence of meaning between, say, a_{m+1}^{ν} and a_{m+1}^{t} should cause no problem since almost always the affixes will be literally 't' or 'ν'.

[31]This number represents the 'density of money' in the poorest sector of the economy – poorest, that is, in terms of ready cash. It corresponds to the derivative at the origin of the well-known *Lorenz curve*, commonly used to depict the distribution of wealth throughout a population; see e,g, Atkinson (1970).

[32]Since we have normalized $\mu(T)=1$, this ratio is a direct comparison of the monetary endowment of the average trader with that of the (essentially) poorest trader.

(ii) *If Assumption A′ is satisfied, then* $\mu(Q_i^\nu) = 0(1/\alpha_0^\nu)$.

The following lemmas prepare for the proof of Theorem 4.

Lemma 2. Let $\delta > 0$ *be arbitrary, and let* z^ν *be the final allocation and* p^ν *the price vector at an open SE of some game* $\Gamma_i(\mathscr{E}^\nu)$, $i = 1, 2$ *or* 3, *whose utilities satisfy Assumption A with bounding function* P_δ. *Suppose that for some* $j = 1, \ldots, m$ *there is a non-null set* V_j *such that* $z_j^{t\nu} > \delta$ *for all* $t \in V_j$. *Then* $p_j^\nu < P_\delta^t$ *for almost all* $t \in V_j$.

Proof. Apply Assumption A to $x = z^{t\nu} - \Delta e_j$, with 'all sufficiently small $\Delta > 0$' including the added assumption $\Delta < \delta$. Then (6.1) will hold for a.a. t in the non-null set V_j, and such traders t will prefer the bundle

$$y^t = z^{t\nu} - \Delta e_j + P_\delta^t \Delta e_{m+1}$$

to the given bundle z^t. So y^t must be infeasible for almost all $t \in V_j$, since almost all traders are utility maximizers at an open SE and would buy the better bundle if they could. On the other hand, it is clear that the similar bundle

$$\hat{y}^t = z^{t\nu} - \Delta e_j + p_j^\nu \Delta e_{m+1}$$

is feasible. Therefore, the infeasibility of y^t can be due only to y_{m+1}^t being greater than $\hat{y}_{m+1}^t$. Comparing the two expressions displayed above, we see that this means that $p_j^\nu < P_\delta^t$ for a.a. t in V_j, as was to be shown. □

The next lemma is a useful corollary to Lemma 2.

Lemma 3. For any $\delta > 0$, *define the function* f_δ: $[0, \infty) \to [0, 1]$ *by*

$$f_\delta(p_j) = \mu(\{t \in T : P_\delta^t > p_j\}).$$

Then under the hypotheses (and notation) of Lemma 2, $\mu(V_j) \leq f_\delta(p_j^\nu)$.

Proof. Apparent from Lemma 2. □

Our final lemma is designed to do service in section 7 as well as in the present proof. First, some definitions. Given an economy $\mathscr{E}$, we say that a pair (q, r) is *associated to* $\mathscr{E}$ (where q: $T \to \mathbb{R}_+^m$ and r: $T \to \mathbb{R}_+^m$ are measurable maps) if any of the following conditions are satisfied:

(a) $s = (q, r)$ is an open SE of $\Gamma_1(\mathscr{E})$ or $\Gamma_3(\mathscr{E})$,
(b) r alone is an open SE of $\Gamma_2(\mathscr{E})$ and $q_j^t = a_j^t$, a.e.,
(c) $q_j^t = \max\{a_j^t - z_j^t, 0\}$ and $r_j^t = p_j \max\{z_j^t - a_j^t, 0\}$, a.e,

(d) $q^t_j = a^t_j$ and $r^t_j = p_j z^t_j$, a.e.,

for all $j=1,\dots,m$, where (for (c) and (d)) $\langle \tilde{p}, z\rangle$ is a CE of $\mathscr{E}$ with[33] $\tilde{p}_{m+1}=1$. Thus the pair (q,r) may arise from an SE (needed for Theorem 4) or from a CE (needed for Theorem 5), and in either case there will be an accompanying price vector $p \in \Omega^m$.

We also define $Q(q,r) \subset T$ to be *associated to* $\mathscr{E}$ if (q,r) is associated to $\mathscr{E}$ and

$$Q(q,r)=\left\{t\in T\colon \sum_{j=1}^{m} r^t_j \geq a^t_{m+1}\right\}.$$

Lemma 4. Let $\mathscr{E}$ satisfy Assumption A′, and let

$$\bar{P}=\bar{P}_{\delta_0}\geq 0, \quad \text{where} \quad 0<\delta_0<\min_{1\leq j\leq m}\int a_j.$$

Suppose that (q,r) and $Q(q,r)$ are associated to $\mathscr{E}$, with accompanying prices p. Then:

(i) $p_j \leq \bar{P}$ for all $j=1,\dots,m$.
(ii) *A positive constant L exists such that $\mu(Q(q,r))\leq L/\alpha_0$; it depends on $\tilde{\mathscr{E}}$ but not on the money endowments a^t_{m+1}.*

Proof. By assumption, (q,r) (or r alone) is an open SE of one of the $\Gamma_i(\mathscr{E})$ or is derived from a CE of $\mathscr{E}$. Hence, by Lemma 1 and the definition of CE (which excludes $p_j=\infty$), there are in every case prices $p\in\Omega^m$ and allocations $z^t \in \Omega^{m+1}$ for $t\in T$ which correspond to the (q,r) in question.

If $p_j=0$, then (i) is obviously true. If $p_j>0$, then it is evident that $\int a_j = \int z_j$. Consider the non-null set of traders $\{t\in T\colon z^t_j \geq \int a_j\}$, and suppose, contrary to (i), that $p_j > \bar{P}$. Then by Assumption A′ almost any trader t in this set could improve his utility by reducing z^t_j by a small $\Delta>0$ (which we assume to be less than $\int a_j$) and increasing z^t_{m+1} by the amount $p_j\Delta$. This maneuver is available whether we are dealing with an SE or a CE, and contradicts the assumed equilibrium. Thus, Lemma 4(i) is proved.

For (ii), suppose that $\mu(Q(q,r))>0$. Then there is a good $j\neq m+1$ and a subset Q' of $Q(q,r)$ such that

$$r^t_j \geq a^t_{m+1}/m \quad \text{for} \quad \text{all } t\in Q'_j \tag{6.4}$$

and

$$\mu(Q')\geq \mu(Q(q,r))/m. \tag{6.5}$$

Clearly $p_j>0$, since otherwise almost any trader $t\in Q'$ could reduce r^t_j

[33]Recall that $\tilde{p}$ is the $(m+1)$-dimensional vector $(p,\tilde{p}_{m+1})$.

without affecting his consumption of j but increasing his consumption of money, thereby improving his utility and upsetting the assumed equilibrium. But $p_j > 0$ implies that

$$\int r_j / p_j = \int q_j,$$

hence by (6.4) and (6.5) we have

$$p_j = \frac{\int r_j}{\int q_j} \geq \frac{\alpha_0 \mu(Q')}{m^2 \int a_j} \geq \frac{\alpha_0 \mu(Q(q,r))}{m^2 \int a_j}, \tag{6.6}$$

and by Lemma 4(i) and (6.6),

$$\frac{\alpha_0 \mu(Q(q,r))}{m^2 \int a_j} \leq \bar{P},$$

from which Lemma 4(ii) follows by taking L to be the maximum of $m^2 \bar{P} \int a_j$ over $1 \leq j \leq m$, which is clearly independent of a_{m+1} as claimed.[34] □

Proof of Theorem 4. First let $i = 1, 2$ or 3 and consider a sequence of games $\Gamma_i(\mathscr{E}^\nu)$, $\nu = 1, 2, \ldots$, having open SEs $s_i^\nu = (q^\nu, r^\nu)$ [where we set $q^\nu = (a_1, \ldots, a_m)$ if $i = 2$]. We may suppose that $\mu(Q_i^\nu) > 0$. Then there is a good j^ν and a subset $\tilde{Q}_i^\nu \subset Q_i^\nu$ such that

$$r_{j^\nu}^{t\nu} \geq a_{m+1}^{t\nu} / m, \quad \text{a.a. } t \text{ in } \tilde{Q}_i^\nu, \tag{6.7}$$

and

$$\mu(\tilde{Q}_i^\nu) \geq \mu(Q_i^\nu)/m. \tag{6.8}$$

Denote by p^ν and z^ν the prices and final allocation produced at (q^ν, r^ν). By (6.7) and (6.8),

$$p_{j^\nu}^\nu = \frac{\int r_{j^\nu}^\nu}{\int q_{j^\nu}^\nu} \geq \frac{\alpha_0^\nu \mu(Q_i^\nu)}{m^2 \int a_{j^\nu}}. \tag{6.9}$$

Let $S_j = \{t \in T \colon a_j^t \geq \int a_j\}$ (i.e., the set of traders whose supply of good j is at or above average) and define $K = \min_{1 \leq j \leq m} \{\mu(S_j)\} > 0$. For each ν, define

$$S'(\nu) = \{t \in S_{j^\nu} \colon q_{j^\nu}^{t\nu} > a_{j^\nu}^t / 2\} \quad \text{and} \quad S''(\nu) = \{t \in S_{j^\nu} \colon q_{j^\nu}^{t\nu} \leq a_{j^\nu}^t / 2\}$$

which are complements in S_{j^ν}. There are two cases to consider:

[34] The value of a_{m+1} plays an incidental role in the proof, but does not enter even indirectly into the expression for L,

Case A. Assume $\mu(S'(\nu)) \geq \mu(S_{j^\nu})/2 \geq K/2$. [Note that $\Gamma_2(\mathscr{E})$ always falls into this case.] Then

$$\int q^\nu_{j^\nu} \geq \int_{S'(\nu)} q^\nu_{j^\nu} > (K/4) \int a_{j^\nu},$$

from which we get, for a.a. t in $\tilde{Q}^\nu_i$,

$$z^{t\nu}_{j^\nu} \geq \frac{r^{t\nu}_{j^\nu}}{\int a^\nu_{m+1}} \int q^\nu_{j^\nu} \geq \frac{\alpha^\nu_0/m}{\int a^\nu_{m+1}} \frac{K}{4} \int a_{j^\nu}, \tag{6.10}$$

which by hypothesis is bounded away from zero – say, by $\delta_1 > 0$. By Lemma 3 with δ_1 for δ, j^ν for j, and $\tilde{Q}^\nu_i$ for V_j, it follows that

$$\mu(\tilde{Q}^\nu_i) \leq f_{\delta_1}(p^\nu_{j^\nu}). \tag{6.11}$$

Case B. Assume that Case A does not hold. Then

$$\mu(S'(\nu)) < \mu(S_{j^\nu})/2,$$

and so

$$\mu(S''(\nu)) > \mu(S_{j^\nu})/2 \geq K/2. \tag{6.12}$$

By the definition of $S''(\nu)$, we see that for a.a. $t \in S''(\nu)$,

$$\begin{aligned} z^{t\nu}_{j^\nu} \geq a^t_{j^\nu} - q^{t\nu}_{j^\nu} \geq a^t_{j^\nu}/2 \geq \int a_{j^\nu}/2 \geq \min_{1 \leq j \leq m} \{\textstyle\int a_j/2\} \\ = \delta_2, \quad \text{say}. \end{aligned} \tag{6.13}$$

Again, we invoke Lemma 3, this time with δ_2 for δ, j^ν for j, and $S''(\nu)$ for V_j. This gives

$$\mu(S''(\nu)) \leq f_{\delta_2}(p^\nu_{j^\nu}). \tag{6.14}$$

With this preparation, we finish the proof of Theorem 4(i) by a contradiction. If $\mu(Q^\nu_i)$ does not go to 0 with ν then there is a subsequence N of the ν's and a positive number γ such that

$$\mu(Q^\nu_i) > \gamma, \quad \text{all } \nu \in N. \tag{6.15}$$

Observe then that, by (6.3) and (6.9),

$$p^\nu_{j^\nu} \to \infty \quad \text{as} \quad \nu \to \infty \text{ in } N. \tag{6.16}$$

Now partition N into complementary subsequences:

$N_A = \{v \in N$ and Case A applies$\}$, and
$N_B = \{v \in N$ and Case B applies$\}$.

We remark that at least one of these is infinite.

If N_B is infinite, then it is clear that by (6.16) the right-hand side of (6.14) goes to 0 as $v \to \infty$ in N_B, contrary to the assertion in (6.12) that $\mu(S''(v)) > K/2$ for all $v \in N_B$.

If N_A is infinite, then once again by (6.16) the right-hand side of (6.11) goes to 0 as $v \to \infty$ in N_A. However, by (6.8) and (6.15), the left-hand side is bounded from below by $\gamma/m > 0$, a contradiction. This completes the proof of Theorem 4(i).

Finally, Theorem 4(ii) is immediate from Lemma 4(ii). □

7. Remarks and examples

The hypotheses in parts (i) and (ii) of Theorem 4 say that the endowment of money – or its value relative to the other goods – is not too 'skewed' across the traders. If either one of these hypotheses is satisfied, then the measure of the noncompetitive traders becomes arbitrarily small as the total money supply increases, through we have as yet no assurance that an actual CE will be found at the limit (see section 8). But if both hypotheses are violated, then the sequence of SEs need not approach full competitivity, in the sense of the theorem, no matter how much money we pour into the economy.

As in section 4, we shall see that the best way to illustrate our theorem is to violate its hypotheses.

Example 1. Consider a sequence of 'sell-all' market games $\{\Gamma_2(\mathscr{E}^v)\}_{v=1}^{\infty}$ with just one good besides money – i.e., $m=1$. Let the traders T comprise the half-open unit interval, and for each v, let the initial endowments be

$$\begin{cases} a_1^{tv} = 1 & \text{all} \quad t \in (0,1], \\ a_2^{tv} = v^{1+\varepsilon} & \text{if} \quad 0 < t \leq v^{\Delta-\varepsilon}, \\ a_2^{tv} = v & \text{if} \quad v^{\Delta-\varepsilon} < t \leq 1, \end{cases} \tag{7.1}$$

where $\varepsilon > 0$, $\Delta > 0$, $\Delta - \varepsilon < 0$. [See fig. 3(a)]. Note that the ratio $\int a_2^v/\alpha_0^v$ is of the order $v^{1+\Delta}/v$, so the hypothesis of part (i) is here violated. To describe the utility functions, let $\beta > 0$, $\gamma > 0$, be real parameters and define

$$\tilde{u}^t(x_1, x_2) = t^{-\gamma} \int_1^{x_1} y^{-\beta} \, dy + x_2$$

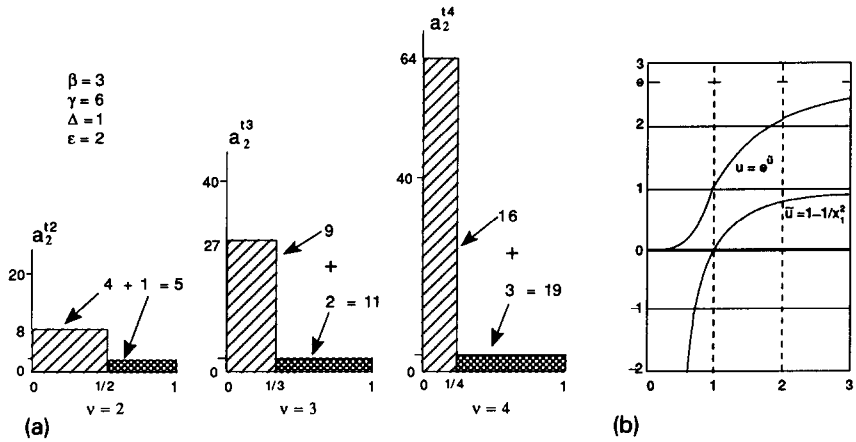

Fig. 3. (a) Initial distributions of money, showing the endowments of the two classes of traders. (b) The utility of money (at $t=0.8909$, $x_2=0$).

for $t\in(0,1]$, $x_1\in(0,\infty)$, $x_2\in[0,\infty)$. (Here the '1' could be any positive number, and $\int_1^{x_1}=-\int_{x_1}^{1}$). Then define $u^t\colon \Omega^2\to\mathbb{R}$ by

$$u^t(x_1,x_2)=\begin{cases} e^{\tilde{u}^t(x_1,x_2)} & \text{if } x_1>0,\\ 0 & \text{if } x_1=0.\end{cases}$$

Note that, for each $t\in(0,1]$, u^t is quasi-concave but *not* concave. [See fig. 3(b)]. Also, denoting $\partial u^t(x)/\partial x_j$ by $u_j^t(x)$ and $\partial\tilde{u}^t(x)/\partial x_j$ by $\tilde{u}_j^t(x)$, $j=1,2$, we have

$$u_1^t(x)/u_2^t(x)=\tilde{u}_1^t(x)/\tilde{u}_2^t(x) \tag{7.2}$$

whenever $x_1>0$. Finally, the $\{u^t\}_{t\in T}$ satisfy Assumption A with $P_\delta^t=1+t^{-\gamma}\delta^{-\beta}$. But they do not satisfy Assumption A′, so the hypothesis of part (ii) is violated. This completes the description of the sequence $\{\Gamma_2(\mathscr{E}^\nu)\}_{\nu=1}^{\infty}$.

Suppose the parameters β, γ, Δ, ε satisfy the inequalities

$$\begin{cases} 1+\Delta+(\Delta-\varepsilon)(\gamma-\beta)<0,\\[2ex] 1+\Delta-\Delta\beta<0,\\[2ex] \Delta-\varepsilon<0.\end{cases} \tag{7.3}$$

The numerical choices $\beta=3$, $\gamma=6$, $\Delta=1$, and $\varepsilon=2$ demonstrate that the system (7.3) is feasible. We claim that the strategies $r^{t\nu}=r_1^{t\nu}=a_2^{t\nu}$, $t\in(0,1]$, constitute an open SE of $\Gamma_2(\mathscr{E}^\nu)$ for all sufficiently large ν.

To check this, first let us compute the price $p^\nu = p_1^\nu$ of good 1 at the above strategy selection. Since everything goes to market, we have

$$p^\nu = \int a_2^{t\nu} / \int a_1^{t\nu} = \nu^{1+\varepsilon} \nu^{\Delta-\varepsilon} + \nu(1-\nu^{\Delta-\varepsilon})$$

[see (7.1)], and hence

$$\nu^{1+\Delta} < p^\nu < 2\nu^{1+\Delta}.$$

So, if $t \in (0, \nu^{\Delta-\varepsilon}]$, then

$$\begin{cases} z_1^{t\nu} = \nu^{1+\varepsilon}/p^\nu < \nu^{1+\varepsilon}/\nu^{1+\Delta} = \nu^{\varepsilon-\Delta}, \\ z_2^{t\nu} = p^\nu; \end{cases}$$

while, if $t \in (\nu^{\Delta-\varepsilon}, 1]$ then

$$\begin{cases} z_1^{t\nu} = \nu/p^\nu < \nu/\nu^{1+\Delta} = \nu^{-\Delta}, \\ z_2^{t\nu} = p^\nu. \end{cases}$$

For any $t \in (0,1]$ and $x \in \Omega^2$ with $x_1 > 0$, let $R^{t\nu}(x)$ denote $u_1^t(x)/u_2^t(x) = \tilde{u}_1^t(x)/\tilde{u}_2^t(x)$ and note that $R^{t\nu}(x) = \tilde{u}_1^t(x)$ since $\tilde{u}_2^t(x) = 1$. Now, if $t \in (0, \nu^{\Delta-\varepsilon}]$, then for sufficiently large ν, say $\nu > \nu_1$, we have

$$\begin{aligned} R^{t\nu}(z^{t\nu}) &= \frac{1}{t^\gamma} \frac{1}{(z_1^{t\nu})^\beta} \geq \frac{1}{\nu^{(\Delta-\varepsilon)\gamma}} \frac{1}{\nu^{(\varepsilon-\Delta)\beta}} \\ &= \nu^{(\varepsilon-\Delta)(\gamma-\beta)} > 2\nu^{1+\Delta} > p^\nu. \end{aligned} \tag{7.4}$$

The first '>' here sets the limit on ν_1 with the aid of the first '>' of (7.3). [For the numerical parameter values suggested at (7.3) one can take $\nu_1 = 1$]. On the other hand if $t \in (\nu^{\Delta-\varepsilon}, 1]$, then for sufficiently large ν, say $\nu > \nu_2$, we have

$$\begin{aligned} R^{t\nu}(z^{t\nu}) &= \frac{1}{t^\alpha} \frac{1}{(z_1^{t\nu})^\beta} \geqq \frac{1}{1} \frac{1}{\nu^{-\Delta\beta}} \\ &> 2\nu^{1+\Delta} > p^\nu. \end{aligned} \tag{7.5}$$

The first '>' in this case sets the limit on ν_2, using the second '>' of (7.3). (Again we may take $\nu_2 = 1$ in the numerical case).

From (7.4) and (7.5) we have obtained that for all $\nu > \max\{\nu_1, \nu_2\}$, $R^{t\nu}(z^{t\nu}) > p^\nu$ for all traders t, which implies they would all spend more if they could. So we are at an SE. Moreover the set of non-interior traders is $(0,1]$, so $\mu(Q_2^\nu) = 1$ for all $\nu > \max\{\nu_1, \nu_2\}$ and the conclusion of Theorem 4 follows.

Example 2. Similar examples for the games Γ_i, $i=1$ and 3, can be constructed by adding another commodity and doubling the player set, as follows. Indeed, let $m=2$ and $T=(0,2]$. The endowments of the non-money goods are given by

$$a_1^t=\begin{cases}1 & \text{for} \quad t\in(0,1],\\ 0 & \text{for} \quad t\in(1,2],\end{cases}$$

$$a_2^t=\begin{cases}0 & \text{for} \quad t\in(0,1],\\ 1 & \text{for} \quad t\in(1,2],\end{cases}$$

and of the money good by

$$a_3^t=\begin{cases}v^{1+\varepsilon} & \text{for} \quad t\in(0,v^{\Delta-\varepsilon}]\cup(1,1+v^{\Delta-\varepsilon}],\\ v & \text{for} \quad t\in(v^{\Delta-\varepsilon},1]\cup(1+v^{\Delta-\varepsilon},2].\end{cases}$$

For the utility functions, define

$$\tilde{u}^t(x_1,x_2,x_3)=\begin{cases}u^t(x_2,x_3) & \text{for} \quad t\in(0,1],\\ u^t(x_1,x_3) & \text{for} \quad t\in(1,2],\end{cases}$$

where $u^t:\Omega^2\to\mathbb{R}$ is defined for $t\in(0,1]$ exactly as in Example 1.

Thus each trader likes only the good he does not start with, and between money and this good his utility is as before, once we identify the points t in (0, 1] with the points $1+t$ in $(1,2]$. This completes the specification of the sequence of economies $\{\mathscr{E}^v\}_{v=1}^{\infty}$ and hence the associated market games Γ_1^v and Γ_3^v.

Now consider the strategy selection wherein each trader puts up all his goods for sale at the appropriate trading post and sends all his money to the other trading post. We submit that for all $v>\max\{v_1,v_2\}$ this selection constitutes an open SE of $\Gamma_i(\mathscr{E}^v)$ for $i=1$ or 3; the argument is exactly the same as in Example 1. □

Discussion of Example 2. Since $\mu(Q_i^v)=2$ (for $i=1$ and 3, and large enough v), almost all traders are 'liquidity-constrained' at the above SE. The question is how severely does the constraint bear on them. We have yet to devise a measure of this severity (for any player at any given strategy selection in our games). But, armed with such a measure, it would be easier to discuss how 'far' the SEs are from being competitive. If, for a given ε, most traders are eventually liquid-constrained up to *at most* the 'level ε' and for large enough v, then the SE may be viewed to be ε-close to competitive – this in spite of the fact that the $\mu(Q_i^v)$ stay large. On the other hand, if there is a $K\gg0$ such that eventually most traders are constrained up to *at least* the 'level K', then the SE are 'K-far' from competitive. This view is consistent with the notion already used in Theorem 4, where the $\mu(Q_i^\nu)\to 0$, i.e., most traders are not liquidity-constrained at all, so that we do not need to worry about what the measure is for them.

For better focus, we consider the numerical case $\beta=3$, $\gamma=6$, $\Delta=1$, $\varepsilon=2$. Let $\tilde{z}^{t\nu}$ denote the (unique) optimal bundle of trader $t\in(0,2]$ in his 'CE budget set' calculated with respect to the SE price vector $\tilde{p}^\nu\equiv(\pi^\nu,\pi^\nu,1)$, where $\pi^\nu=\nu^2+\nu-1$ is the common price of goods 1 and 2 in terms of good 3 at the SE of $\Gamma_i(\mathscr{E}^\nu)$, $i=1,3$. Thus $\tilde{z}^{t\nu}$ is the unique maximizer of u^t over the set

$$\{x\in\Omega^3: \tilde{p}^\nu x\leq\tilde{p}^\nu a^{t\nu}\}.$$

Trader t would demand $\tilde{z}^{t\nu}$ were he to behave as in the CE world, facing prices $\tilde{p}^\nu$.

It will also be useful at this point, for our purposes, to think of an imaginary extension $\tilde{\Gamma}_i^\nu$ of the games $\Gamma_i(\mathscr{E}^\nu)$, for $i=1,3$, in which each t can borrow, before trade and at zero interest, as much of good 3 as he likes. If he fails to repay the loan after trade, then his payoff in $\tilde{\Gamma}_i^\nu$ is some given numbers less than $u^t(a^{t\nu})$. Otherwise it is the utility of his consumption. It is clear that, with the price of good 3 normalized to 1, the SE outcomes of $\tilde{\Gamma}_i^\nu$ coincide with the CE outcomes of $\mathscr{E}^\nu$.

We observe that the SE of $\Gamma_i(\mathscr{E}^\nu)$ that we have calculated are *feasible* in $\tilde{\Gamma}_i^\nu$ but do not constitute an SE of $\tilde{\Gamma}_i^\nu$. This is because each trader t would have an incentive to deviate by borrowing and spending more of good 3. How much more? The optimal amount, as called for in his best response to $\tilde{p}^\nu$ in $\tilde{\Gamma}_i^\nu$ – or better, this amount divided by t's money endowment $a_{m+1}^{t\nu}$ – may be taken to indicate how 'thirsty for money' (i.e., severely liquidity-constrained) he is at the SE of $\Gamma_i(\mathscr{E}^\nu)$ in question.

To examine this, it will help first to calculate the individual excess demand (in the CE sense) of trader t for good j, namely the quantity

$$\tilde{z}_j^{t\nu}-a_j^{t\nu}. \tag{7.6}$$

Denote

$$t^\nu\equiv(\nu^2+\nu-1)^{1/3}(\nu^2+2\nu-1)^{-1/2}$$

and

$$r^\nu\equiv(\nu^2+\nu-1)^{1/3}(\nu^2+\nu-1+\nu^3)^{-1/2}.$$

For large enough ν, $t^\nu\approx\nu^{-1/3}>\nu^{-1}$ and $r^\nu\approx\nu^{-5/6}>\nu^{-1}$, and for such ν we obtain, by a straightforward computation, the results given in table 1.

The *aggregate* excess demand for good j is

Table 1

Individual excess demand.

For t in...	Good 1	Good 2	Good 3
$(0,\nu^{-1}]$	-1	$1+\nu^3(\pi^\nu)^{-1}$	$-\nu^3$
(ν^{-1},t^ν)	-1	$1+\nu(\pi^\nu)^{-1}$	$-\nu$
$(t^\nu,1]$	-1	$t^{-2}(\pi^\nu)^{-1/3}$	$\pi^\nu-t^{-2}(\pi^\nu)^{2/3}$
$(1,1+\nu^{-1}]$	$1+\nu^3(\pi^\nu)^{-1}$	-1	$-\nu^3$
$(1+\nu^{-1},1+t^\nu]$	$1+\nu(\pi^\nu)^{-1}$	-1	$-\nu$
$(1+t^\nu,2]$	$t^{-2}(\pi^\nu)^{-1/3}$	-1	$\pi^\nu-t^{-2}(\pi^\nu)^{2/3}$

$$\int_0^2 (\bar{z}_j^{t\nu}-a_j^{t\nu})\,\mathrm{d}t, \tag{7.7}$$

by (7.6). It is positive for $j=1$ and 2 and goes to 0 as $\nu\to\infty$. For $j=3$, however, it is negative and diverges to $-\infty$ as $\nu\to\infty$. But once we scale it down by dividing by the aggregate endowment $\int_0^2 a_3^{t\nu}\,\mathrm{d}t$, it too goes to 0.

The agregate picture, however, misses the enormous thirst for money that exists at the individual level. Indeed, since $t^\nu\to 0$ as $\nu\to\infty$, the overwhelming majority of the traders are in the set

$$(t^\nu,1]\cup(1+t^\nu,2].$$

They wish to spend $\pi^\nu t^{-2}(\pi^\nu)^{-1/3}=t^{-2}(\pi^\nu)^{2/3}$ units of good 3 (money) to purchase their favored non-money good, and so wish to borrow

$$\max\{0,t^{-2}(\pi^\nu)^{2/3}-\nu\}\equiv\beta^{t\nu},\quad \text{(say)}.$$

Define, for any $K>0$,

$$B_K^\nu=\{t\colon \beta^{t\nu}/a_3^{t\nu}>K\}.$$

Then, remembering that $(\pi^\nu)^{2/3}\approx\nu^{4/3}$ and that $a_3^{t\nu}=\nu$ for $t\in(t^\nu,1]\cup(1+t^\nu,2]$, we see that for any $K>0$ there exists $\nu(K)$ such that if $\nu>\nu(K)$ then

$$\lambda(B_K^\nu)>1-\frac{1}{K},$$

where λ denotes the Lebesgue measure.

Thus, most traders become unboundedly liquidity-constrained in our sequence of SEs.

Monotonicity of the money supply. If we pick the subsequence $\{\mathscr{E}^{\nu_i}\}_{i=1}^\infty$ so that $\nu_{i+1}\geq\nu_i^{1+\varepsilon}$ for all i, then $a_3^{t\nu_i}$ is increasing in i for each t in $[0,1]$. Thus a requirement that the money be put in monotonically for each trader does nothing to stop the examples.

Concavity v. quasi-concavity. Our examples are based on utility functions that are quasi-concave but not concave. We have not found a way of adapting our examples to concave utility functions, not do we know if the assumption of concavity, in conjunction with Assumption A, would enable us to dispense with the special hypotheses in (i) and (ii) and still obtain the key conclusion of Theorem 4 that $\mu(Q_i^\nu) \to 0$ as $\nu \to \infty$.

Remark on the existence of SEs. We cannot infer the existence of SEs of $\Gamma_i(\mathscr{E})$, for $i=1,2,3$, from the existence of CEs of $\mathscr{E}$, since the two are not equivalent in general. Thus, the issue of the existence of SEs becomes important in the commodity-money models.

Let us say that t 'likes' j if u^t is strictly increasing in the jth variable and 'has' j if $a_j^t > 0$. Our conjecture is as follows:

Let a be integrable, u measurable, and u^t nondecreasing, continuous, and quasi-concave for all $t \in T$. Suppose further that for each $j=1,\dots,m$ there is (i) a non-null set of traders who have money and like j, and (ii) a non-null set of traders who have j and like money. Then for $i=1,3$ an open SE exists for $\Gamma_i(\mathscr{E})$ with all prices positive and finite, while for $i=2$ the same holds without condition (ii).

While we have not checked the details, an analogous result has been provided[35] for the case of a finite number of traders, with 'non-null set' in the above replaced by 'at least two'.

8. Exact equivalence

Although Theorem 4 shows that the SE solutions become more and more nearly competitive as the money supply is increased, it falls short of asserting the actual coincidence of the CE and SE allocations. No matter how much money is pumped into the economy $\tilde{\mathscr{E}}$, Assumption A still permits a small band of 'fanatics' to persist whose unusual endowments or tastes cause them to hit the spending limit and want to spend more. Further conditions can be adduced, however, that exclude this possibility. The easiest is the familiar simplifying assumption that there are only finitely many *types* of traders, but we shall first consider other conditions. Recall...

Assumption A′. For each $\delta > 0$ there is a number $\bar{P}_\delta > 0$ such that, for each $j=1,\dots,m$,

[35] Dubey and Shubik (1978).

$$u^t(x+\Delta \bar{P}_\delta e_{m+1}) > u^t(x+\Delta e_j)$$

holds a.e. for all sufficiently small $\Delta > 0$ and all $x \in \Omega^{m+1}$ with $x_j \geq \delta$.

This says in essence that the marginal utility of money relative to that of the other goods always 'holds up' – uniformly across the traders – unless they happen to be near zero levels in their consumption of those goods. We previously used this assumption in Theorem 4, where it was enough to establish an explicit rate of convergence for $\mu(Q_i^\nu)$, $i=1,2,3$. But we shall require further conditions to get the exact equivalence we now seek.

Assumption B. For each $\varepsilon > 0$ there is a number $G_\varepsilon > 0$ such that, for each $j=1,\dots,m$,

$$u^t(x+\varepsilon\Delta e_{m+1}) > u^t(x+\Delta e_j)$$

holds a.e. for all sufficiently small $\Delta > 0$, and all $x \in \Omega^{m+1}$ with $x_j \geq G_\varepsilon$.

This assumption further differentiates the goods from money. It says that there is a tendency towards satiation in each good, relative to money, as the traders get swamped by that good.

Our final assumption is that for every good, there is a non-null set of trader who like it so well that they are never wholly satiated:

Assumption C. For each $j=1,\dots,m$, there is a non-null set T_j with the property that given any $\alpha > 0$, a number $\delta_j(\alpha) > 0$ exists such that

$$u^t(x+\delta_j(\alpha)\Delta e_{m+1}) < u^t(x+\Delta e_j)$$

holds a.e. in T_j for all sufficiently small $\Delta > 0$ and for all $x \in \Omega^{m+1}$ with $\max_{1 \leq l \leq m} x_l < \alpha$.

Theorem 5. Let the sequence $\{\mathscr{E}^\nu = (\tilde{\mathscr{E}}, a_{m+1}^\nu)\}_{\nu=1}^\infty$ satisfy (6.3), and let the u^t satisfy Assumptions A′, B, C. Then there is a $\tilde{\nu}$ such that, if $\nu \geqq \tilde{\nu}$, the CE prices and allocations (and hence payoffs) of $\mathscr{E}^\nu$ coincide with those achieved at the open SEs of $\Gamma_i(\mathscr{E}^\nu)$, for $i=1,2,3$.

Proof. To prove the theorem, it will suffice to show that there is a $\tilde{\nu}$ such that $\mu(Q(q^\nu, r^\nu)) = 0$ for all $\nu \geq \tilde{\nu}$, for all $Q(q^\nu, r^\nu)$ associated to $\mathscr{E}^\nu$.

First observe that, by Assumption A′ and Lemma 4, there is a constant L depending on $\tilde{\mathscr{E}}$ such that

$$\mu(Q(q^\nu, r^\nu)) \leq L/\alpha_0^\nu.$$

Let p^ν and z^ν be the prices and allocation at (q^ν, r^ν). Let T_j be as in Assumption C and define, for any ν, and $\beta > 0$, and $1 \leq j \leq m$, $1 \leq l \leq m$,

$$T_j(\nu, \beta, l) = \{t \in T_j \colon z_l^{t\nu} \geq \beta\}.$$

Then

$$\mu(T_j(\nu, \beta, l)) \leqq \int a_l/\beta.$$

Denote $\min\{\mu(T_j)\colon j = 1, \ldots, m\}$ by M, and choose β^* and γ^* sufficiently large to ensure that

$$M > \sum_{j=1}^{m} \frac{\int a_j}{\beta^*} + \frac{L}{\gamma^*}.$$

Next choose ν^* to ensure that $\alpha_0^\nu > \gamma^*$ whenever $\nu > \nu^*$. Then, if we let

$$T_j^\nu = \left\{ t \in T_j \colon z_l^{t\nu} < \beta^* \quad \text{for} \quad l = 1, \ldots, m, \sum_{l=1}^{m} r_l^{t\nu} < a_{m+1}^{t\nu} \right\},$$

it follows that

$$T_j^\nu \supset T_j \setminus \left(Q(q^\nu, r^\nu) \cup \bigcup_{l=1}^{m} T_j(\nu, \beta^*, l) \right),$$

hence

$$\mu(T_j^\nu) > M - \sum_{l=1}^{m} \frac{\int a_l}{\beta^*} - \frac{L}{\gamma^*} > 0 \tag{8.1}$$

for all $\nu > \nu^*$ and $j = 1, \ldots, m$.

Now, by Assumption C and the definition of T_j^ν, we have

$$u^t(z^{t\nu} + \delta_j(\beta^*)\Delta e_{m+1}) < u^t(z^{t\nu} + \Delta e_j) \tag{8.2}$$

for sufficiently small Δ, all $\nu > \nu^*$, and a.a. $t \in T_j^\nu$. If $p_j^\nu < \delta_j(\beta^*)$ for some $\nu > \nu^*$, then since $\sum_{j=1}^{m} r_j^{t\nu} < a_{m+1}^{t\nu}$ for a.a. $t \in T_j^\nu$, almost every trader in T_j^ν could improve his payoff by buying a bit more of good j. But by (8.1), T_j^ν is non-null for all $\nu > \nu^*$, which contradicts the assumption that (q^ν, r^ν) is associated to $\mathscr{E}^\nu$. So we must have

$$p_j^\nu \geq \delta_j(\alpha^*) > 0 \tag{8.3}$$

for all $\nu > \nu^*$. On the other hand, by Lemma 4 (which applies because Assumption A′ has been made), we also have

$$p_j^\nu \leq L \tag{8.4}$$

for all ν, where L is provided by Lemma 4.

Suppose there is a subsequence $\bar{N}$ of the ν's such that $\mu(Q(q^\nu, r^\nu)) > 0$ for all ν in $\bar{N}$. Then there exists a good $1 \leq j \leq m$ and a further subsequence $\bar{\bar{N}}$ such that, for each $\nu \in \bar{\bar{N}}$, we can find a subset S^ν of $Q(q^\nu, r^\nu)$ with

$$\mu(S^\nu) > 0 \tag{8.5}$$

and

$$r_j^{t\nu} \geq a_{m+1}^{t\nu}/m \quad \text{for} \quad \text{a.a. } t \in S^\nu. \tag{8.6}$$

But, from (8.4) and (8.6) we obtain, for all $\nu \in \bar{\bar{N}}$ and a.a. $t \in S^\nu$,

$$z_j^{t\nu} \geq \frac{r_j^{t\nu}}{p_j^\nu} \geq \frac{a_{m+1}^{t\nu}}{mL} \geq \frac{\alpha_0^\nu}{mL} \to \infty. \tag{8.7}$$

Finally, take $\{\varepsilon_n : n = 1, 2, \ldots\}$ such that $\varepsilon_n \to 0$. By (8.7), we can find a subsequence $\{\nu_n\} \subset \bar{\bar{N}}$ with $z_j^{t\nu_n} > G_{\varepsilon_n}$ for all n and almost all $t \in S^\nu$, where G_{ε_n} is as in Assumption B. Therefore, by Assumption B,

$$u^t(z^{t\nu_n} + \varepsilon_n \Delta e_{m+1}) > u^t(z^{t\nu_n} + \Delta e_j) \tag{8.8}$$

for all sufficiently small Δ, almost all $t \in S^{\nu_n}$ and all $n = 1, 2, \ldots$. If $p_j^{\nu_n} > \varepsilon_n$, then by (8.8) almost every trader in S^{ν_n} could unilaterally improve his payoff by selling a little bit of good j. Since $\mu(S^\nu) > 0$ by (8.5), this contradicts the assumption that (q^ν, r^ν) is associated to $\mathscr{E}^{\nu_n}$. So

$$p_j^{\nu_n} < \varepsilon_n \to 0. \tag{8.9}$$

But (8.9) contradicts (8.3), so we conclude that there is a $\hat{\nu}$ such that $\mu(Q(q^\nu, r^\nu)) = 0$ for all $\nu \geq \hat{\nu}$. $\square$

Finite type economies. Let us call $\mathscr{E}$ a *finite type economy* if its trader set T admits a finite partition $\{T_1, \ldots, T_\tau\}$ such that any two traders in the same T_i have identical endowments and utility functions. It is clear that at any SE of $\Gamma(\mathscr{E})$ the members of a given T_i are facing exactly the same optimization problem; moreover almost all of them are solving it – though not necessarily in the same way. This means that either a.a. $t \in T_i$ are competitive or a.a. $t \in T_i$ are not competitive, since either the maximum utility in $B^t(\tilde{p})$ [see (5.8)] is attainable without exceeding the spending limit or it is not. It follows that the measure $\mu(Q)$ of the set of all noncompetitive traders at the SE must be some partial sum of $\{\mu(T_i),\ i = 1, \ldots, \tau\}$. This indicates that if $\mu(Q)$ is small, it must be 0. Formally, *Theorem 6. Consider a class of finite type economies $(\tilde{\mathscr{E}}, a_{m+1})$, differing only in their money endowments α_{m+1} and admitting a common type partition $\{T_1, \ldots, T_\tau\}$. Let the utilities satisfy Assumption A′. Then there is a bound $D = D(\tilde{\mathscr{E}})$ such that if $\alpha_0 \geq D$ then:*

(i) *The prices and allocations (and hence, payoffs) of any open SE of $\Gamma_i(\tilde{\mathscr{E}}, a_{m+1})$ are competitive, for $i = 1, 2, 3$.*
(ii) *The prices and payoffs of any CE of $(\tilde{\mathscr{E}}, a_{m+1})$ are achieved at some open SE of $\Gamma_i(\tilde{\mathscr{E}}, a_{m+1})$, for $i = 1, 2, 3$.*

Proof. Let (q, r) be an open SE of $\Gamma_i(\tilde{\mathscr{E}}, a_{m+1})$ for $i = 1, 2$ or 3. By Lemma 4,

$$\mu(Q(q, r)) \leq L/\alpha_0,$$

where L depends on $\tilde{\mathscr{E}}$ but not on a_{m+1}. Let $\mu_0 = \min_{1 \leq i \leq \tau} \mu(T_i)$ and $D = L/\mu_0$. If $\alpha_0 > D$, the only way the inequality displayed above can hold is if $\mu(Q(q, r)) < \mu_0$. Hence, in each type $i = 1, \ldots, \tau$, there is a non-null set of traders who are interior and who (by Lemma 1) achieve their maximum utility in $B^i(\tilde{p}) \equiv B^t(\tilde{p})$ for $t \in T_i$. But all players in T_i face the same optimization problem at the SE. So almost all of them must have achieved their maximum utility in $B^i(\tilde{p})$. This being true for $i = 1, \ldots, \tau$, we have proved (i) of Theorem 6.

To prove (ii), let $\langle \tilde{p}, z \rangle$ be the prices and allocation at a CE of $(\tilde{\mathscr{E}}, a_{m+1})$ with $\tilde{p} \in \Omega^{m+1}$ and $\tilde{p}_{m+1} = 1$. Define the allocation $\tilde{z}$ by

$$\hat{z}^t = \int_{T_i} z^t \, \mathrm{d}\mu / \mu(T_i), \quad \text{if} \quad t \in T_i,$$

for $1 \leqq i \leqq \tau$. It is readily checked that $\langle \tilde{p}, z \rangle$ is also a CE of $(\tilde{\mathscr{E}}, a_{m+1})$; and by the quasi-concavity of u^t, we have $u^t(\hat{z}^t) = u^t(z^t)$ for almost all t. Define

$$\hat{r}^t_j = \tilde{p}_j \max\{z^t_j - a^t_j, 0\},$$

$$\hat{q}^t_j = \max\{a^t_j - z^t_j, 0\}$$

for $1 \leq j \leq m$. Then $(\hat{q}, \hat{r})$ is associated to $\mathscr{E}$, and

$$\mu(Q(\hat{q}, \hat{r})) \leq L/\alpha_0 < \mu_0.$$

(The $\leq$ follows from Lemma 4; the $<$ holds because $L = \mu_0 D$ by definition and $D < \alpha_0$ by assumption). But from our construction, $Q(\hat{q}, \hat{r})$ must be one of the sets $T_1, \ldots, T_\tau$ or else the empty set. Since each $\mu(T_i) \geq \mu_0$, $Q(\hat{q}, \hat{r})$ is the empty set. This shows that $(\hat{q}, \hat{r})$ is an open SE of $\Gamma_1(\tilde{\mathscr{E}}, a_{m+1})$ and $\Gamma_3(\tilde{\mathscr{E}}, a_{m+1})$.

For $\Gamma_2(\tilde{\mathscr{E}}, a_{m+1})$, set $\hat{q}^t_j = a^t_j$ and $\hat{r}^t_j = \tilde{p}_j z^t_j$ and repeat the above argument. □

Appendix: Non-atomic games in strategic form

In this final section, we turn to the problem of building a satisfactory connection between the strategic decisions of the individual players in a non-atomic game and the events that take place in the model-at-large as a result of those decisions. The principal source of difficulty lies in the conflict between the 'noncooperative' ideal of strictly independent decision-making and the technical demands of measurability and integrability. Although we sidestepped this issue in the body of the paper, we are unwilling to walk away from it. Some readers may agree that the problem is real, and the rather intricate approach that we suggest below is appropriate; others may only find in this discussion merely a renewed reason for wanting to evade the issue – or, better, an incentive for seeking alternative resolutions.[36]

In order to focus on the main question we shall make the simplifying assumption that the players' strategy spaces are all the same, say, $\Sigma^t \equiv [0, 1]$. A strategy selection is then *any* function $g\colon T \to [0, 1]$. Let $\mathscr{G}$ denote the set of all such functions and $\mathscr{M}$ the subset of those that are measurable. While g may represent the declared intentions of the individual players, to implement these intentions in the market as a whole will require passing from g to a set function $G\colon \mathscr{C} \to \mathbb{R}$ – hopefully by simple integration:

$$G(S) = \int_S g(t)\,\mathrm{d}\mu(t), \quad \text{all} \quad S \in \mathscr{C}. \tag{A.1}$$

But this step is not directly permissible unless $g \in \mathscr{M}$. What if $g \in \mathscr{G} \backslash \mathscr{M}$?

We begin with the observation that any set function $g \in \mathscr{G}$ can be 'trapped' or 'squeezed' between two measurable functions, say $g_*, g^* \in \mathscr{M}$, in such a way that the value of the integral $\int_T [g^*(t) - g_*(t)]\,\mathrm{d}\mu$ is minimized subject to $g_*(t) \le g(t) \le g^*(t)$ a.e.[37] Moreover, these minimizing functions are uniquely determined a.e., so the corresponding set functions G_* and G^* [cf. (A.1)] are uniquely determined and are of course countably additive, being bounded by the functions 0 and μ. The function G that we seek will presumably be found

[36]Bierlein (1981) (which contains reference to earlier work), compares our present construction to a promising method of *measure extension* that seeks out 'measurable neighbors' of non-measurable functions. Unfortunately not all non-measurable strategy selections possess measurable neighbors, as there defined.

[37]The reader will note the similarity with the inner and outer measures of a not-necessarily-measurable set.

somewhere between G_* and G^*. But there are many such functions, and to define a particular G will require further specification.

At this point we could make an arbitrary modeller's decison and, for example, write

$$G = cG_* + (1-c)G^*, \tag{A.2}$$

where c is a constant between 0 and 1. Without further motivation, however, this is unsatisfactory. Accordingly, we shall devise a method of extending the underlying strategic model that has some heuristic substance. A formula like (A.2) might be the result of such an extension, but other results might perhaps be more suitable in a given application.

Our basic idea is to put the job of ensuring integrability into the hands of the players. Since this will require some sort of collective action, we introduce the notion of a 'group' strategy. The members of a set $S \in \mathscr{C}$ get together and jointly declare the total amount, say $\phi(S)$, that they expect to send to market. This will be a number between 0 and $\mu(S)$.[38] In keeping with the noncooperative canon, we take these joint declarations to be all independent, i.e., not constrained by the declarations of other, possibly overlapping groups. Moreover, we shall require declarations only from very small groups, since only the limiting properties of ϕ on very small sets will matter in the end. We thereby keep to a minimum the intrusion of 'cooperative' action into the extended model. No group of positive measure, or set of groups whose measure is bounded away from 0, can affect the result by their declarations. If a 'group-decision theory' – by which we mean a mapping from the individual-intention functions g to the group-declaration functions ϕ – can be found that behaves well in the limit, then a mode of determinate play based on arbitrary (possibly non-measurable) strategy selections will have been achieved.

We may imagine that there is a referee or game-master who can hear only measurable instructions. So the players band together in order to make their moves audible. But they do not band together to play the game cooperatively in the usual sense.

Let us now turn attention to the passage to the limit. There is no reason to suppose that the set function ϕ will be additive. But it is possible that, if we look at its values only on very small sets, it will be 'nearly' additive. If $S \in \mathscr{C}$ and $\delta > 0$, let $\mathscr{P} = \mathscr{P}(S, \delta)$ denote the set of *finite partitions* $\Pi = \{R_1, \ldots, R_p\} \subset \mathscr{C}$ of S into nonempty sets of measure $0 \leq \mu(R_j) \leq \delta$. Partially-

[38]Note that $\mu(S)$ is the total amount available to S in our simplified set-up. We do not require a priori that $\phi(S)$ lie between G_* and G^*, though it is not implausible that a reasonable group-decision theory would confine ϕ within that interval.

order $\mathscr{P}$ by refinement.[39] Then $(\mathscr{P}, \leq)$ is a *directed set*,[40] and we can define a set function ϕ_0, called the *additive part* of ϕ, by

$$\phi_0(S) = \operatorname*{dir\,lim}_{\Pi \in \mathscr{P}} \sum_{R \in \Pi} \phi(R), \tag{A.3}$$

under the assumption that these directed limits exist for all $S \in \mathscr{C}$. If by chance ϕ was already additive, then of course $\phi_0 = \phi$. More generally, ϕ_0 is the additive function that is 'tangent' to ϕ, in the sense that it most closely approximates ϕ on small sets.

One way to assure that a set function possesses an additive part is to require that it be superadditive or subadditive, or more generally, that its *total deviation*, be bounded.[41] That all such functions have additive parts follows easily from the observation that the inequalities.

$$0 \leq \sum_{R' \in \Pi'} \phi(R') \leq \sum_{R \in \Pi} \phi(R), \quad \text{all} \quad \Pi' \leq \Pi \in \mathscr{P},$$

which hold for every nononegative, superadditive ϕ, ensure the monotonic convergence of the directed limit (A.3).

Examples. We now give some examples of group decision rules that lead to coalitional strategy selections that are of bounded deviation.

One simple rule is the following:

$$\phi(S) = \mu(S) \inf g(S). \tag{A.4}$$

In other words, the group takes its cue from its most timid member. Alternatively, the group could ignore 'timid' sets of measure zero and replace 'inf' by 'ess inf'.[42] In either case, ϕ is obviously subadditive and hence (since $\phi - \mu$ is nonpositive) a member of BD. When we take the additive part of ϕ the distinction disappears, since during the refinement sequence we can

[39] I.e., $\Pi \geq \Pi'$ means that each element of Π is a subset of some element of Π'.

[40] That is, every finite subset of $\mathscr{P}$ has a lower bound in $\mathscr{P}$ [See Dunford and Schwartz (1958, p. 45)]. By the *directed limit* of any $f: 2^{\mathscr{C}} \to \mathbb{R}$ is meant the number f_0 (unique if it exists) such that for every $\varepsilon > 0$ there is a Π_ε in $\mathscr{P}$ such that $\Pi < \Pi_\varepsilon$ implies $|f(\Pi) - f_0| < \varepsilon$.

[41] Shapley (1953), Armstrong (1991). The *total deviation* of a set function ϕ is most simply defined as the minimum of $g(T) - h(T)$ where g and h are nonnegative and superadditive and $\phi - (g - h)$ is additive. At the minimizing g, h the latter is precisely ϕ_0. The set functions of bounded deviation form a Banach space *BD*, in which total deviation is a *pseudonorm*, i.e., a norm on the subspace defined by $\phi_0 = 0$. [Cf. the rather similar notion of *bounded variation* employed by Aumann and Shapley (1974).]

[42] See section 6 at (6.3).

always split off any null sets that cause 'inf' and 'ess inf' to be different. In fact, it is not difficult to show that $\phi_0 = G_*$, in each case.

The 'bold' rule analogous to (A.4) leads similarly to $\phi_0 = G^*$, or we might suppose that groups divide the range between these two extremes in some definite ratio:

$$\phi(S) = \mu(S)[c \inf g(S) + (1-c) \sup g(S)].$$

where $0 < c < 1$. The leads of course to the formula (A.2).

A more interesting possibility is the following. Let α: $T \to [0,1]$ be a fixed measurable function, and define

$$\phi = \int_S \operatorname{med}\,[\inf g(S), \sup g(S), \alpha(t)]\, \mathrm{d}\mu(t), \tag{A.5}$$

where $\operatorname{med}[x, y, z]$ denotes the median of x, y and z. Perhaps $\alpha(t)$ might represent a generally held 'preconceived idea' of what player t will choose, before his actual intention $g(t)$ becomes known. The spokesman for the group S, unable to process all the disorderly information contained in the non-measurable function g, bases his declaration instead on the preconceptions $\alpha(t)$, except where they are inconsistent with the more easily observed inf and sup of $g(S)$.

Thus, let

$$S_1 = \{t \in S : \alpha(t) < \inf g(S)\},$$

$$S_2 = \{t \in S : \alpha(t) > \sup g(S)\},$$

$$S_3 = S \backslash S_1 \backslash S_2.$$

Since these sets are measurable, we may rewrite (A.5) as

$$\phi(S) = \mu(S_1) \inf g(S) + \mu(S_2) \sup g(S) + \int_{S_3} \alpha(t)\, \mathrm{d}\mu(t).$$

This function can be shown to be in BD,[43] and its additive part is given by

$$\phi_0(S) = \int_S \operatorname{med}\,[g_*(t), g^*(t), \alpha(t)]\, \mathrm{d}\mu(t),$$

[43]It is neither subadditive nor superadditive in general, but its total deviation can be shown to be $\leq \mu(T)[\sup g(T) - \inf g(T)]$.

which lies between G_* and G^* as expected, but is not a simple average as in (A.2). Note that if $g \in \mathscr{M}$ then α drops out of the calculation.

These examples, though based on strategy sets simpler than those in the economic models treated in this paper, suffice to show that by including a small touch of cooperative detail in the model, the strategic form of a non-atomic game can be made 'complete', in the sense that arbitrary choices by the individual players always lead to a definite and reasonable outcome, and the 'group decision rule' adopted for the completion will not affect the measurable SEs of the game.

Moreover, without going into further detail at this time, we may remark that for certain classes of utility functions the best-response mapping applied to *any* selection produces a measurable selection. This means that non-measurable SEs are not possible. (Cf. the discussion in section 3).

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