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INCENTIVE COMPATIBILITY IN A MARKET WITH INDIVISIBLE GOODS *

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In a market in which each trader's initial endowment is one unit of an indivisible good, there exists an incentive compatible procedure for reaching a competitive allocation. This contrasts with some recent results for similar problems.

1. Introduction

This note shows that in a market where each trader's initial endowment is one unit of an indivisible good, there is a procedure for reaching a competitive allocation that makes it a dominant strategy for each player to reveal his true preferences over all goods in the market. Shapley and Scarf (1974) showed that a competitive allocation always exists in such a market. Roth and Postlewaite (1977) showed that when no trader is indifferent between any goods, this competitive allocation is unique. These results can be proved using a constructive procedure, called the method of top trading cycles, which Shapley and Scarf attribute to David Gale. The procedure studied here is a variant of Gale's method of top trading cycles.

2. The model

Consider a market with n traders, each of whom owns one indivisible good. (Shapley and Scarf suggest a market in houses as an example.) The traders have ordinal preferences over the goods.

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Let the initial endowment be $w=(w_1, \ldots, w_n)$, where w_i is the good brought to market by the ith trader. Denote the ith trader's transitive preference relation by R_i, where $w_j R_i w_k$ means trader i likes w_j at least as well as w_k. If $w_j R_i w_k$ but not $w_k R_i w_j$, trader i strictly prefers w_j to w_k; denote this by $w_j P_i w_k$.

An allocation is any permutation of the initial endowment w. The set of allocations represents the set of all trades which result in each trader having possession of exactly one item. An allocation will sometimes be denoted as a vector $x=(x_1, \ldots, x_n)$, where the x_i can be mapped by some one-to-one mapping into the corresponding w_j.

3. The procedure

Since players may report indifference as well as strict preference between goods, the allocation procedure will employ an arbitrary, but fixed, method of resolving ties. The discussion will proceed as if each trader i reported a strict preference relation P_i reflecting no indifference: this may be thought of as the strict order which results after any indifferences reported by i are resolved (e.g., ties might be broken by alphabetical order).

Let N be the index set of all traders and goods, and $P=(P_1, \ldots, P_n)$ be a strict preference for each trader. For any subset N' of N, define the directed graph $G'(P)=(N', A(N',P))$ whose nodes are N' and whose (directed) arcs $A(N',P)$ consist of the elements (i,j) such that i, j are in N' and trader i prefers w_j to all other w_k with k in N', i.e., $w_j P_i w_k$ for all $k \neq i$ in N'. The graph G' is the graph on N' which results when each trader in N' points to his favorite good in N'. A *cycle* is a sequence of nodes $S=(n_1, n_2, \ldots, n_k=n_0)$ for which (n_q, n_{q+1}) is an arc of the graph for $q=0, \ldots, k$. A cycle may contain only a single node, when some trader i prefers w_i to all other goods on the market N'. The *trade corresponding to a cycle S* is defined to be the trade which gives trader n_q the initial endowment of n_{q+1} for $q=0, \ldots, k$. For any preference profile $P=(P_1, \ldots, P_n)$, the top trading cycle procedure can now be described as follows:

1 (a) At time $t=1$, each trader i points to his most preferred w_j, resulting in the graph $G_1(P)=(N_1, A(N_1, P))$, with $N_1=N$.

(b) In each cycle of $G_1(P)$ the corresponding trade is performed: the members of each cycle (traders and their endowments) are then removed from the market. If any traders remain in the market, the procedure goes on to the next step.

⋮

k (a) At time $t = \mathrm{k}$, each trader i remaining in the market (after cycles have been removed from G_{k-1}) points to his most preferred w_j remaining in the market, resulting in the graph $G_k(P) = (N_k, A(N_k, P))$.

(b) In each cycle of $G_k(P)$ the corresponding trade is performed, and the members of each cycle are removed from the market. The procedure goes on to step $k+1$ if any traders remain in the market, otherwise it terminates.

This procedure terminates in at most n steps, since the finiteness of N implies that at least one cycle forms at each step. The resulting allocation is the one in which each trader receives the good assigned to him by the trade corresponding to the cycle which removed him from the market. To see that this allocation is competitive, let all goods which left the market at step k have the same price, and let the prices of goods from different steps be higher for goods leaving earlier. Any such prices are competitive, since players find the most preferred good in their budget set in the cycle in which they were removed from the market. The main result of this paper can now be formally stated.

Theorem. In the procedure described above, it is a dominant strategy for each player to reveal his true preferences.

4. Proofs

In what follows, let $P = (P_1, \ldots, P_n)$ be some fixed preference profile, in which any indifferences reported by the traders have been resolved by some fixed tie-breaking procedure. The specific realization of the top trading cycle procedure when P reflects the reported preferences will be denoted $T(P)$, the grapth at the kth stage of $T(P)$ will be $G_k(P)$, and the resulting allocation will be $t(P)$. Let P' be a preference profile which differs from P only in the report of a single (fixed) trader i, who reports P_i' instead of P_i. To prove the theorem, it will be sufficient to show that if $x = t(P)$ and $y = t(P')$, then no P_i' exists for which $y_i P_i x_i$: that is, if P_i is trader i's true preference, he cannot get a preferred outcome by reporting P_i' instead of P_i.

Lemma 1. *If $S=(n_1, n_2, \ldots, n_m)$ is a chain in $G_k(P)$ [i.e., if (n_q, n_{q+1}) is an arc of $G_k(P)$ for $q=1, \ldots, m-1$] and if $r>k$, then S is a chain in $G_r(P)$ if and only if n_m is a node of $G_r(P)$.*

Proof. The proof by induction follows immediately from the observation that if (n_{m-1}, n_m) is an arc of $G_k(P)$ then it is an arc of $G_r(P)$ if and only if n_m is a node of $G_r(P)$.

Lemma 2. *Let P and P' be as above, and let k and k' be the periods at which trader i is removed from the market in $T(P)$ and $T(P')$, respectively. Then the graphs $G_l(P)$ and $G_l(P')$ have the same cycles for $1 \leqslant l \leqslant \min\{k, k'\}-1$, and the same nodes for $1 \leqslant l \leqslant \min\{k, k'\}$.*

Proof. N is the set of nodes for both $G_1(P)$ and $G_1(P')$. Since these two graphs differ only in the arc emanating from node i, they have the same cycles if $\min\{k, k'\}>1$. In this case $G_2(P)$ has the same set of nodes as $G_2(P')$ and the lemma follows by induction.

Lemma 3. *Let P'' be a preference profile which differs from P' only in the report of trader i, where P_i'' is any preference such that $y_i P_i'' y_j$ for all $j \neq i$. Then if $z=t(P'')$, $z_i=y_i$.*

Proof. Let k' be the period of $T(P')$ at which i and y_i are removed from the market, and let $w_j=y_i$; i.e., y_i is the initial endowment of trader j. Lemma 2 implies that, if i is still on the market at period k' of $T(P'')$, then $G_{k'}(P')=G_{k'}(P'')$, since both graphs contain the arc (i,j). In this case, therefore, i is matched with y_i at step k' of $T(P'')$, so $z_i=y_i$. But Lemmas 1 and 2 also imply that y_i cannot be removed from the market prior to trader i, so i must be assigned y_i on or before period k' of $T(P'')$.

Proof of the Theorem. Let P and P' be as above, with $x=t(P)$, $y=t(P')$, and $y_i=w_j$. Let k and k' be the periods of $T(P)$ and $T(P')$, respectively, at which trader i is removed from the market. We will assume that either $y_i P_i x_i$ or $y_i=x_i$, and then show that $y_i=x_i$. Lemma 3 implies it is sufficient to consider a preference P_i' that ranks y_i first: i.e., we can assume $y_i P_i' y_j$ for all $j \neq i$. If $k' \geqslant k$, then, by Lemma 2, $G_l(P)$ and $G_l(P')$ have the same nodes for $1 \leqslant l \leqslant k$. If $y_i P_i x_i$, Lemma 1 implies that the arc emanating from i in $G_k(P)$ could not terminate in the node corresponding to x_i, since j, the node corresponding to y_i, is one of the nodes of $G_k(P)$. Therefore $y_i=x_i$ [and $k' \geqslant k$ implies $G_k(P')=G_k(P)$, so $k'=k$].

Thus $k' \leqslant k$, and so $G_l(P)$ and $G_l(P')$ have the same nodes for $1 \leqslant l \leqslant k'$. Let $S = (j = n_1, n_2, \ldots, n_m = i)$ be the cycle that forms at step k' of $T(P')$. Except possibly for the arc (i,j) all the other arcs of S are contained in $G_{k'}(P)$ as well as $G_{k'}(P')$, since P and P' differ only in the ith component. So S forms a chain in $G_{k'}(P)$, and, by Lemma 1, S is also a chain in $G_k(P)$. Once again, if $y_i P_i x_i$, then Lemma 1 implies the arc emanating from i could not terminate at the node corresponding to x_i, since j is a node of $G_k(P)$. So $y_i = x_i$.

5. Concluding remarks

Since the competitive allocation is in the core of the market considered here, we can compare the above theorem with results concerning incentive compatible procedures for reaching the core in similar problems whose outcomes also consist of permutations of the set of agents. One such problem is the matching, or 'marriage' problem, considered by Gale and Shapley (1962), which involves two disjoint sets of agents, each having preferences over the opposite set, in which outcomes match agents from one set with agents from the other. In the context of the 'market for houses' considered here, matching can be thought of as a restriction to bilateral trades: if trader i is assigned w_j, then j must be assigned w_i.

Roth (1982) shows that no incentive compatible procedure exists for reaching a point in the (always non-empty) core of the matching problem, but that procedures for reaching the core do exist which give all the agents in *one* of the two sets the incentive to correctly reveal their preferences. Ritz (1982) shows this latter result continues to hold for substantially generalized versions of the matching problem.

Intermediate between the housing market and the matching problem is the 'roommate problem', in which there is only one set of agents, each of whom has preferences over the rest. Outcomes match agents bilaterally. In this problem, the core need not even be non-empty.

So the matching problem, which has two sets of agents and bilateral trades, has a non-empty core which cannot be reached by any incentive compatible procedure. Relaxing the constraint that agents from one set must be matched with agents from the other yields the roommate problem, in which the core can be empty. Additionally relaxing the constraint that trades must be bilateral yields the housing market studied here, in which the core is non-empty and can be reached by an incentive compatible procedure. It might be tempting to try to draw an analogy between the two sets of agents in the matching problem and the sets of agents and houses in the housing market, but this does not seem to go through. But is seems likely that there are some general regularities, as yet unknown, which unify these disparate results.

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